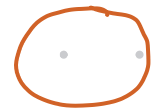
16 April 2024

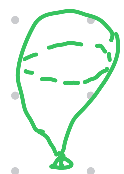
BETTI NUMBERS OF A SIMPLICIAL COMPLEX

β_0 = number of connected components (0-dimensional holes)

β_1 = number of 1-dimensional holes



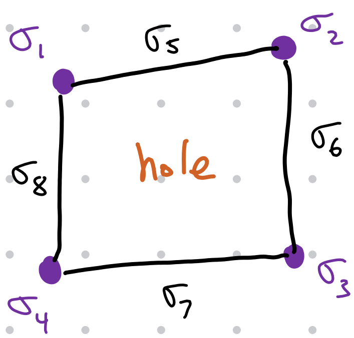
β_2 = number of 2-dimensional holes



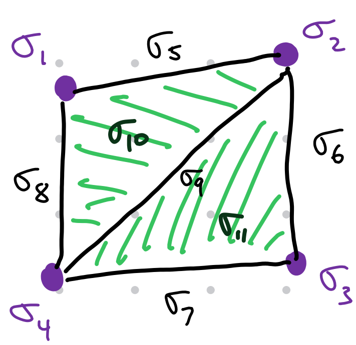
⋮ etc.

How do we compute the Betti numbers?

EXAMPLE:



This hole is identified by simplices $\sigma_5, \sigma_6, \sigma_7$, and σ_8 forming a <u>cycle</u> (a loop) with nothing inside.



If we fill in the hole, then $\sigma_5, \sigma_6, \sigma_7$, and σ_8 still form a cycle, but now it is the boundary of simplices σ_{10} and σ_{11}.

KEY IDEA #1.

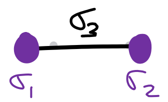
A <u>hole</u> is a <u>cycle</u> that is not a <u>boundary</u>.

→ a ~~set~~ sum of simplices with no boundary.

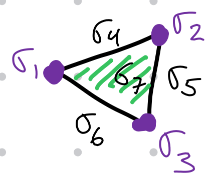
DEFINITIONS:

∂ d-dimensional

The **boundary** of a d-simplex is the sum of its (d-1)-dimensional faces.

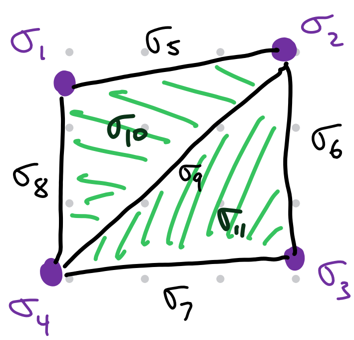
e.g.: 

$$\partial(\sigma_3) = \sigma_1 + \sigma_2$$



$$\partial(\sigma_7) = \sigma_4 + \sigma_5 + \sigma_6$$

The **boundary of a sum** of simplices is the sum of the boundaries, reduced mod 2.

e.g. 

$$\partial(\sigma_5 + \sigma_6) = \underbrace{\sigma_1 + \sigma_2}_{\partial(\sigma_5)} + \underbrace{\sigma_2 + \sigma_3}_{\partial(\sigma_6)}$$

$$= 1\sigma_1 + 2\sigma_2 + 1\sigma_3 = \sigma_1 + \sigma_3$$

$$\partial(\underbrace{\sigma_5 + \sigma_6 + \sigma_7 + \sigma_8}_{\text{these simplices form a cycle, thus have no boundary}}) = \sigma_1 + \sigma_2 + \sigma_2 + \sigma_3 + \sigma_3 + \sigma_4 + \sigma_4 + \sigma_1$$

$$= 0$$

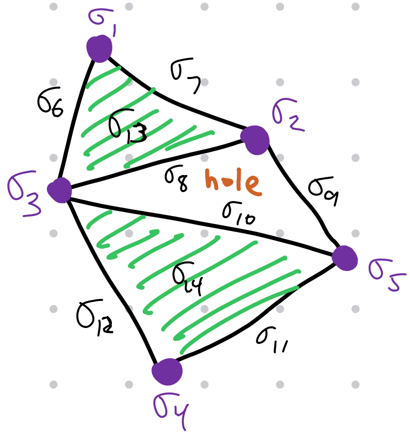
Here, $\sigma_5 + \sigma_6 + \sigma_7 + \sigma_8$ is a cycle and also a boundary, so it is not a hole.

If it was a cycle but not a boundary, then it would be a hole.

$$\partial(\sigma_{10} + \sigma_{11}) = (\sigma_5 + \sigma_8 + \sigma_9) + (\sigma_6 + \sigma_7 + \sigma_9)$$

$$= \underbrace{\sigma_5 + \sigma_6 + \sigma_7 + \sigma_8}_{\text{this is the boundary of the two triangles}}$$

EXAMPLE:



How many holes?

How many cycles that are not boundaries?

We can identify multiple cycles around this hole

e.g.: $A = \sigma_8 + \sigma_9 + \sigma_{10}$ $\qquad$ $B = \sigma_8 + \sigma_9 + \sigma_{11} + \sigma_{12}$

What is the difference between A and B?

$$A + \underline{\ ?\ } = B$$

$$? = \sigma_{10} + \sigma_{11} + \sigma_{12}$$

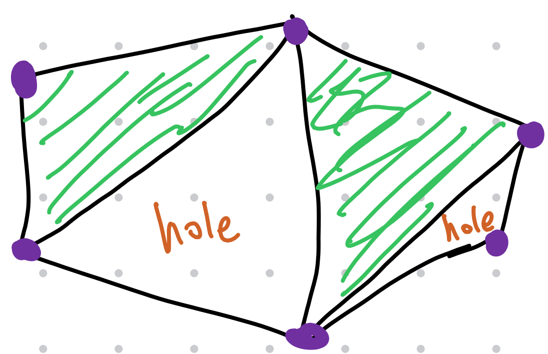
$$A - B = \sigma_{10} + \sigma_{11} + \sigma_{12} = \partial(\sigma_{14})$$

KEY IDEA #2:

Two cycles that differ by a boundary are the same!
homologous

To count holes: identify cycles that do not merely differ by a boundary.

EXAMPLE:



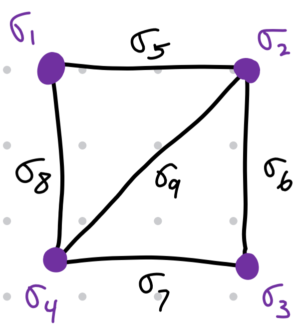
Here, there are exactly 2 families (equivalence classes) of homologous cycles.

How to compute cycles and boundaries?

BOUNDARY MATRIX:

matrix ∂_j encodes the boundaries of all j-dimensional simplices of a simplicial complex K

EXAMPLE:



∂_2 is empty

$$\partial_1 = \begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\partial_0 = [\ \sigma_1 \ \ \sigma_2 \ \ \sigma_3 \ \ \sigma_4\ ] \text{ — 0 rows}$$

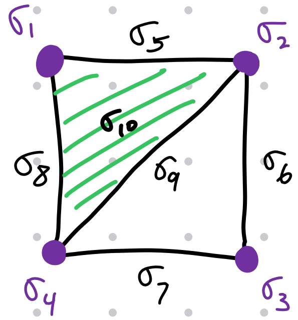
$$\beta_1 = \text{nullity}(\partial_1) - \text{rank}(\partial_2)$$
$$= (5 - \text{rank}(\partial_1)) - 0 =$$
$$= 5 - 3 = 2$$

DEFINITION of Betti numbers:

$$\beta_j(K) = \underbrace{\text{nullity}(\partial_j)}_{\text{number of cycles}} - \underbrace{\text{rank}(\partial_{j+1})}_{\text{that are not boundaries}}$$

$$\beta_j(K) = \big((\text{number of columns in } \partial_j) - \text{rank}(\partial_j)\big) - \text{rank}(\partial_{j+1})$$

$\boxed{\begin{array}{l}\text{RANK-NULLITY THEOREM:} \\ \quad \text{for a matrix } M\text{, the number of columns of } M \\ \quad \text{equals } \text{rank}(M) + \text{nullity}(M). \end{array}}$



$$\partial_2 = \begin{array}{c} \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \end{array} \begin{array}{c} \sigma_{10} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

$$\beta_1 = \Big((\text{num. columns in } \partial_1) - \text{rank}(\partial_1) \Big) - \text{rank}(\partial_2)$$

$$= 5 - 3 - 1$$

$$= 1$$ = number of holes

$$\beta_0 = \text{nullity } \partial_0 - \text{rank } \partial_1$$

$$= 4 - 3 = 1$$ = number of connected components