15 Sept. 2025

Topology Day 2

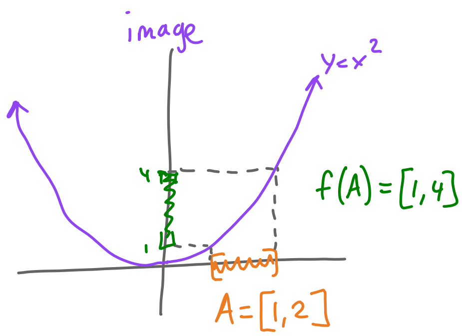
DEFINITIONS

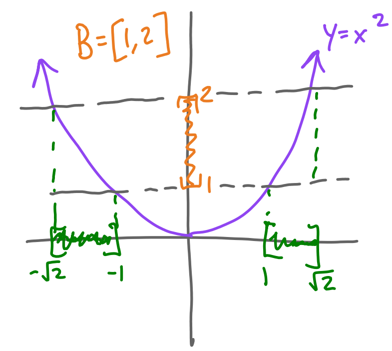
Let $\underset{\text{domain}}{f: D} \to \underset{\text{codomain}}{C}$ be a function.

For any subset $A \subset D$, let $f(A) = \{f(x) \mid x \in A\}$ denotes the **image** of A under f.

For any subset $B \subset C$, let $f^{-1}(B) = \{x \in D \mid f(x) \in B\}$ denote the **preimage** of B under f.

Example: $f(x) = x^2$





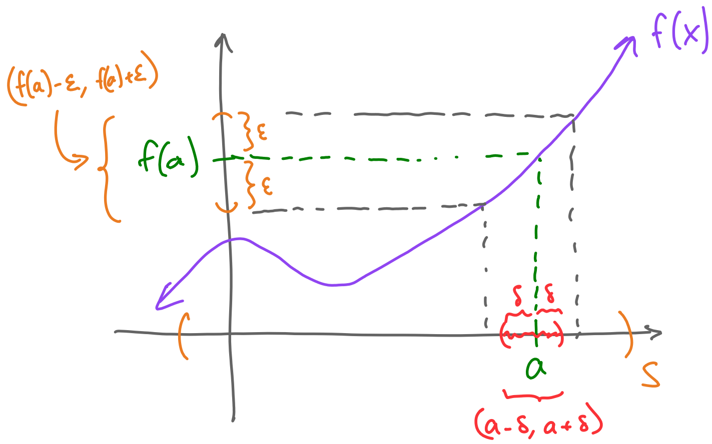
$f^{-1}(B) = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

Continuity from Real Analysis class:

Given any subset $S \subset \mathbb{R}$, a function $f: S \to \mathbb{R}$ is **continuous at a point** $a \in S$ if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $\underbrace{|f(x) - f(a)| < \varepsilon \text{ whenever } |x-a| < \delta}$.

this is: $|x-a| < \delta$ implies that $|f(x) - f(a)| < \varepsilon$

A function $f: S \to \mathbb{R}$ is **continuous** if it is continuous at every point in S.



If $x \in (a-\delta, a+\delta)$

meaning $|a-x| < \delta$,

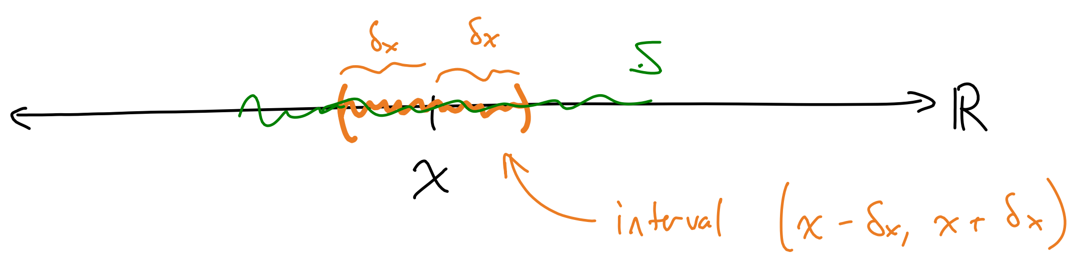
then $f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$

meaning $|f(a)-f(x)| < \varepsilon$.

OPEN SETS

An **open interval** is a set $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$.

A set $S \subset \mathbb{R}$ is **open** if for every point $x \in S$ there is some open interval $(x-\delta_x, x+\delta_x) \subset S$ with $\delta_x > 0$.



Example:

$$S = \bigcup_{i=1}^{\infty} \left(\frac{1}{2i+1}, \frac{1}{2i}\right) = \underset{\uparrow\, i=1}{\left(\frac{1}{3}, \frac{1}{2}\right)} \cup \underset{\uparrow\, i=2}{\left(\frac{1}{5}, \frac{1}{4}\right)} \cup \underset{\uparrow\, i=3}{\left(\frac{1}{7}, \frac{1}{6}\right)} \cup \cdots$$