Generalized Functions

Math 330

1. Let $g_n(x) = \dfrac{n}{\pi(1+n^2x^2)}$ for $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

(a) Make some plots of $g_n(x)$ for various positive integers n. How does the shape of the graph depend on n?

(b) If $a \neq 0$, what is $\lim_{n\to\infty} g_n(a)$? What is $\lim_{n\to\infty} g_n(0)$?

(c) What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\ dx$?

(d) What function is equal to $\lim_{n\to\infty} g_n(x)$?

2. Choose your favorite continuous function $u(x)$. (Everyone at your table should choose a different function.) Explore

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)u(x)\ dx.$$

What do you observe?

3. Let $\sigma_\xi(x) = \int_a^x \delta_\xi(t)\, dt$? Sketch a graph of $\sigma_\xi(x)$.

4. Let $\rho_\xi(x) = \int_a^x \sigma_\xi(t)\, dt$? Sketch a graph of $\rho_\xi(x)$.

5. What is $\dfrac{d\sigma_\xi}{dx}$?

6. Let $f(x) = \begin{cases} -x, & x < 1 \\ x^2, & x > 1. \end{cases}$

(a) Write $f(x)$ as the sum of a continuous function $g(x)$ plus a step function σ_ξ.

(b) Differentiate $f(x)$ in the context of generalized functions.

7. Let $f(x) = \begin{cases} x, & -1 < x < 0 \\ x^2, & 0 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$

Write $f(x)$ as the sum of a continuous function and step functions. Then differentiate $f(x)$ in the context of generalized functions.

8. Does $\delta(x)$ have a Fourier series?

(a) Find the Fourier coefficients of $\delta(x)$.

(b) Does the Fourier series you found in part (a) converge to $\delta(x)$? (Plot some partial sums.)

9. Let $s_n(x) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{n} \cos(kx)$.

(a) What is $\displaystyle\int_{-\pi}^{\pi} s_n(x)\ dx$?

(b) Explore $\displaystyle\lim_{n\to\infty} \int_{-\pi}^{\pi} s_n(x)u(x)\ dx$ for your choice of continuous functions $u(x)$. What do you observe?