Stability Analysis

Math 330

We have observed that the *explicit scheme* for finding approxiate solutions to the heat equation sometimes produces solutions that exhibit crazy behavior over time. To understand this, we will examine the effect of the explicit scheme on simple functions.

Recall that the explicit scheme uses the partial difference equation:

$$u_{j+1,m} = \mu\, u_{j,m+1} + (1-2\mu)u_{j,m} + \mu\, u_{j,m-1}$$

1. Suppose that, at some time t_j, the approximate solution is given by $u(t_j, x) = e^{ikx}$, for some $k \in \mathbb{R}$. Substitute $u(t_j, x) = e^{ikx}$ into the right-hand side of the partial difference equation:

$$u_{j+1,m} =$$

2. Remembering that $x_{m-1} = x_m - \Delta x$ and $x_{m+1} = x_m + \Delta x$, manipulate your equation from #1 to obtain

$$u_{j+1,m} = u_{j,m}\left[1 - 2\mu + \mu\left(e^{ik\Delta x} + e^{-ik\Delta x}\right)\right].$$

3. Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to rewrite your equation as

$$u_{j+1,m} = u_{j,m}\left[1 - 2\mu\left(1 - \cos(k\Delta x)\right)\right].$$

4. Let $\lambda = 1 - 2\mu\,(1 - \cos(k\Delta x))$. The partial difference equation then becomes

$$u_{j+1,m} = \lambda u_{j,m}.$$

What happens to our approximation if $|\lambda| < 1$? What happens if $|\lambda| > 1$?

5. Explain why $\lambda \leq 1$.

6. Show that $-1 < \lambda$ is equivalent to $\dfrac{1}{1 - \cos(k\Delta x)} > \mu$. You may assume $\cos(k\Delta x) \neq 1$.

7. Explain why $-1 < \lambda$ if $\mu < \frac{1}{2}$.

8. Why might the solution be badly behaved if $\mu > \frac{1}{2}$?

9. If we require that $\mu < \frac{1}{2}$, what restrictions does this impose on Δx and Δt?