Laplace's Equation in a Rectangle
Math 330

1. First, we should get a little more comfortable with hyperbolic sine and cosine, denoted $\sinh x$ and $\cosh x$ (hyperbolic sine is pronounced "synch"). They are defined

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Compute $\cosh 0$ and $\sinh 0$.

(b) Compute $\frac{d}{dx} \cosh x$ and $\frac{d}{dx} \sinh x$.

(c) Compute $\cosh^2 x - \sinh^2 x$.

(d) Write e^x and e^{-x} in terms of $\cosh x$ and $\sinh x$. This shows that linear combinations of e^x and e^{-x} can alternatively be written as linear combinations of $\cosh x$ and $\sinh x$.

Now let's discuss Laplace's equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

on the rectangular domain $D = \{(x, y) \mid 0 < x < L, 0 < y < H\}$. The solution $u(x, y)$ gives the equilibrium temperature distribution on the rectangle with prescribed boundary conditions. To keep things simple, we will assume inhomogeneous boundary conditions on only one side and homogeneous boundary conditions on the other three sides:

$$u(x, 0) = 0 \qquad\qquad\qquad u(0, y) = 0$$
$$u(x, H) = 0 \qquad\qquad\qquad u(L, y) = 10$$

2. Let $u(x, y) = v(x)w(y)$. Plug this into the PDE, separate variables, set the resulting expressions equal to a constant λ, and arrive at two ODEs. What are the appropriate boundary conditions for each ODE?

3. One of your ODEs should have two boundary conditions. *This is your eigenvalue problem.* Find the eigenvalues and associated eigenfunctions.

4. Use the eigenvalues you found in the previous problem to solve the second ODE, with the single boundary condition. *Hint*: Express the general solution for this ODE in terms of sinh and cosh, then apply the boundary condition.

5. You should now have solutions to $v(x)$ and $w(y)$, dependent on some set of eigenvalues (indexed by n). Confirm that you get

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right)$$

when you take an infinite sum of these product solutions.

6. Now use orthogonality and the boundary condition $u(L, y) = 10$ to compute the c_n coefficients. This completes the solution for $u(x, y)$. Plot your solution to a sufficient number of terms to check that it looks reasonable.