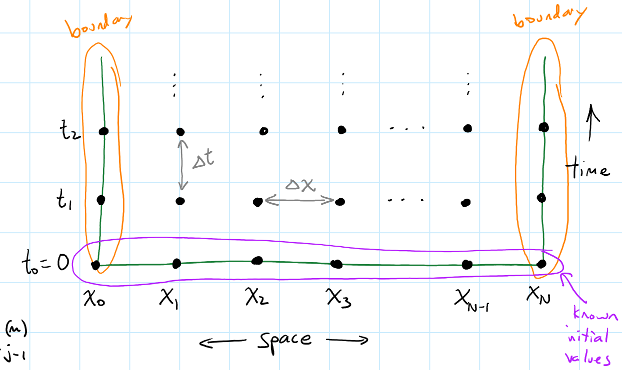
NUMERICAL SOLUTIONS TO THE HEAT EQUATION



Explicit Scheme:

$$u_j^{(m+1)} = s u_{j+1}^{(m)} + (1-2s) u_j^{(m)} + s u_{j-1}^{(m)}$$

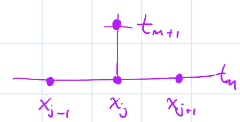
where $u_j^{(m)}$ is the numerical solution at x_j, t_m

The explicit scheme is stable if $s < \frac{1}{2}$ and unstable if $s > \frac{1}{2}$.

$s = \frac{k \Delta t}{\Delta x^2}$, so we want $\frac{k\Delta t}{\Delta x^2} < \frac{1}{2}$, so $\Delta t < \frac{\Delta x^2}{2k}$

The time step must be much smaller than the spatial step.

OBSERVE: $$u_j^{(m+1)} = \underbrace{s u_{j+1}^{(m)} + (1-2s) u_j^{(m)} + s u_{j-1}^{(m)}}_{\text{This is a weighted average of numerical solution values at time } t_m}$$



If $s > \frac{1}{2}$, then the coefficient $1-2s < 0$, which is concerning.

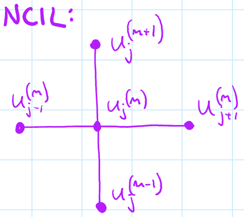
RICHARDSON'S SCHEME

Use centered differences for space and time.

$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ becomes

$$\frac{u(x, t+\Delta t) - u(x, t-\Delta t)}{2\Delta t} = k \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2}$$

STENCIL:



$$\frac{u_j^{(m+1)} - u_j^{(m-1)}}{2\Delta t} = k \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{\Delta x^2}$$

$$u_j^{(m+1)} = u_j^{(m-1)} + \frac{2k\Delta t}{\Delta x^2}\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right)$$

STABILITY: let $s = \frac{k\Delta t}{\Delta x^2}$, $u_j^{(m)} = e^{i\alpha x_j} Q^m$.

Then: $$e^{i\alpha x_j} Q^{m+1} = e^{i\alpha x_j} Q^{m-1} + 2s\left(e^{i\alpha(x_j+\Delta x)} - 2e^{i\alpha x_j} + e^{i\alpha(x_j - \Delta x)}\right) Q^m$$

$$\cancel{e^{i\alpha x_j}} Q^{m+1} = \cancel{e^{i\alpha x_j}} Q^{m-1} + 2s\cancel{e^{i\alpha x_j}}\left(e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x}\right) Q^m$$

$$Q^2 = 1 + 2sQ\left(\underbrace{e^{i\alpha\Delta x} + e^{-i\alpha\Delta x}}_{\cos(\alpha\Delta x) + i\cancel{\sin(\alpha\Delta x)} + \cos(\alpha\Delta x) - i\cancel{\sin(\alpha\Delta x)}} - 2\right)$$

$$Q^2 = 1 + 2sQ\left(2\cos(\alpha\Delta x) - 2\right)$$

$$Q^2 + 4sQ\left(1 - \cos(\alpha\Delta x)\right) - 1 = 0$$

Let $w = 2s(1-\cos(\alpha \Delta x))$.
Observe $w \geq 0$.

$$Q^2 + 2wQ - 1 = 0$$

by the quadratic formula: $Q = \dfrac{-2w \pm \sqrt{(2w)^2 - 4(-1)}}{2} = -w \pm \sqrt{w^2+1}$

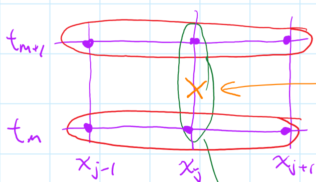
Since $w \geq 0$, we have $w^2+1 \geq 1$, and $-\sqrt{w^2+1} \leq -1$.

Thus, $Q_- = -w - \sqrt{w^2+1} < -1$, so the scheme is <u>unstable for all s.</u>

(Stability requires all Q to satisfy $|Q| \leq 1$.)

CRANK-NICOLSON SCHEME

Take the average of two centered differences for $\frac{\partial^2 u}{\partial x^2}$.



"center" of the approximation

suppose that $\dfrac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$ approximates $\frac{\partial u}{\partial t}\left(x, t + \frac{\Delta t}{2}\right)$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

average of second derivative at two points

forward difference

$$\frac{\partial u}{\partial t}\left(x_j, t_m + \tfrac{\Delta t}{2}\right) = k\, \frac{\frac{\partial^2 u}{\partial x^2}(x_j, t_m) + \frac{\partial^2 u}{\partial x^2}(x_j, t_m + \Delta t)}{2}$$

centered differences

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{2} = \frac{k}{2}\left(\frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{\Delta x^2} + \frac{u_{j+1}^{(m+1)} - 2u_j^{(m+1)} + 2u_{j-1}^{(m+1)}}{\Delta x^2} \right)$$

Let $s = \dfrac{k \Delta t}{\Delta x^2}$ as before. Then:

$$u_j^{(m+1)} - u_j^{(m)} = \frac{s}{2}\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)} + u_{j+1}^{(m+1)} - 2u_j^{(m+1)} + u_{j-1}^{(m+1)} \right)$$

Separate the time indexes: $m+1$ on the left, m on the right.

$$-\frac{s}{2} u_{j+1}^{(m+1)} + (1+s)\, u_j^{(m+1)} - \frac{s}{2} u_{j-1}^{(m+1)} = \frac{s}{2} u_{j+1}^{(m)} + (1-s)\, u_j^{(m)} + \frac{s}{2} u_{j-1}^{(m)}$$

We can write this equation in matrix form and use it to compute the numerical solution.