Last time: solving $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ with $u(0,t)=0$, $u(L,t)=0$

Supposed: $u(x,t) = \phi(x) G(t) \Rightarrow \frac{dG}{dt} = -\lambda k G$ and $\underbrace{\frac{d^2\phi}{dx^2} = -\lambda \phi}$

$\Downarrow$

$G(t) = c e^{-\lambda k t}$

Boundary Value Problem (BVP): $\phi'' = -\lambda \phi$, $\phi(0) = \phi(L) = 0$

If $\lambda < 0$, then only the trivial solution.

If $\lambda = 0$, " " " " ".

If $\lambda > 0$, then $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

Boundaries: $\phi(0) = 0 \Rightarrow 0 = c_2$ so $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x)$

$\phi(L) = 0 \Rightarrow 0 = c_1 \sin(\sqrt{\lambda}\, L)$

Since we want a nontrivial solution, it must be that

$\sin(\sqrt{\lambda}\, L) = 0$, or $\sqrt{\lambda}\, L = n\pi$ for $n \in \{1,2,3,\ldots\}$

Thus, $\sqrt{\lambda} = \frac{n\pi}{L}$ or $\underbrace{\lambda = \left(\frac{n\pi}{L}\right)^2}$ and $\underbrace{\phi(x) = c_1 \sin\left(\frac{n\pi}{L} x\right)}$.

EIGENVALUES and EIGENFUNCTIONS of the boundary value problem

ANALOGOUS

BVP: $\mathcal{L}(\phi) = -\lambda \phi$

From linear algebra: $A\vec{v} = \lambda \vec{v}$

Solution to Problem I: $u(x,t) = \phi(x) G(t)$

$u(x,t) = c_1 \sin\left(\frac{n\pi}{L} x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$ for $n \in \{1,2,3,\ldots\}$

By the principle of superposition, our most general solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}}$$

Back of worksheet: Problem II

Book exercise 2.3.7.

Homogeneous Neumann boundary conditions: $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$

$G(t)$ is still $G(t) = c e^{-k\lambda t}$

$\phi(x)$: If $\lambda < 0$, then no nontrivial solutions.

If $\lambda = 0$, then we find $\phi(x) = b$ is a nontrivial solution.

If $\lambda > 0$, then $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$$\phi'(x) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x)$$

boundary conditions: $\phi'(0) = 0 \Rightarrow c_1 = 0$

$\phi'(L) = 0 \Rightarrow 0 = -\sqrt{\lambda}\, c_2 \sin(\sqrt{\lambda}\, x)$ so $\lambda = \left(\frac{n\pi}{L}\right)^2$ as before

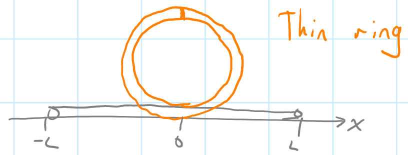
Thus, $u(x,t) = c_2 \cos(\sqrt{\lambda}\, x)\, e^{-k\lambda t}$

Superposition: $$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{\pi n}{L}\right)^2 t}}$$

PROBLEM III:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad u(-L,t) = u(L,t) \text{ and } \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Physical interpretation: rod from $-L$ to L, bent into a circle



Again: $u(x,t) = \phi(x) G(t)$

Still find: $G(t) = e^{-\lambda k t}$

BVP: $\phi'' = -\lambda \phi$ with $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$

If $\lambda < 0$, then only the trivial solution.

If $\lambda = 0$, then $\phi(x) = b$.

If $\lambda > 0$, then $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

again, $\lambda = \left(\frac{n\pi}{L}\right)^2$, but both c_1 and c_2 are arbitrary

Product solution: $\phi(x) = \left(c_1 \sin\left(\frac{n\pi}{L} x\right) + c_2 \cos\left(\frac{n\pi}{L}\right) x\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}$

General solution: $$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}}$$