1. An interviewer is given a long list of people that she can interview. When asked, suppose that each person independently agrees to be interviewed with probability 0.45. The interviewer must conduct ten interviews. Let X be the number of people she must ask to be interviewed in order to obtain ten interviews.

(a) What is the probability that the interviewer will obtain ten interviews by asking no more than 18 people?

X has a negative binomial distribution with $p = 0.45$ and $r = 10$

Then: $P(X \le 18) = \sum_{x=1}^{18} \binom{x-1}{r-1} p^r (1-p)^{x-r} = 0.2527$

(b) What are the expected value and variance of the number of people who decline to be interviewed before the interviewer finds ten people who agree?

Note that $X - 10$ people decline to be interviewed.

$E(X-10) = E(X) - 10 = \frac{10}{0.45} - 10 = 12.22$

$Var(X-10) = Var(X) = \frac{10(1-0.45)}{(0.45)^2} = 27.16$

2. Suppose that $X \sim \text{Exp}(3)$, and let $Y = \lfloor X \rfloor$ denote the largest integer that is less than or equal to X. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 5.99 \rfloor = 5$, and $\lfloor 14 \rfloor = 14$.

(a) Is Y a discrete or continuous random variable?

Possible values of Y are $0, 1, 2, 3, \ldots$, so Y is discrete.

(b) Find $P(Y \le 1)$.

$P(Y \le 1) = P(X < 2) = \int_0^2 3e^{-3x}\,dx = -e^{-3x}\Big|_0^2 = 1 - e^{-6} \approx 0.9975$

(c) Find $P(Y = 2)$.

$P(Y = 2) = P(2 \le X < 3) = \int_2^3 3e^{-3x}\,dx = -e^{-3x}\Big|_2^3 = e^{-6} - e^{-9} \approx 0.0023$

(d) Can you generalize? What is $P(Y = n)$, for any positive integer n? Is the distribution of Y one of the distributions that we have studied in this course?

$$P(Y=n) = P(n \le X < n+1) = \int_n^{n+1} 3e^{-3x}\,dx = -e^{-3x}\Big|_n^{n+1} = -e^{-3(n+1)} + e^{-3n} = e^{-3n}(1-e^{-3}) = (1-p)^n p,$$

where $p = 1 - e^{-3}$.

This is almost the pmf of a geometric random variable.

In fact, $Y+1$ has a geometric distribution with $p = 1-e^{-3}$.

3. Let $X \sim \text{Geom}(p)$. Find the expected value of $\frac{1}{X}$.

X has mass function $p(x) = (1-p)^{x-1}p$ for $x = 1, 2, 3, \ldots$

Then $E\left(\frac{1}{X}\right) = \sum_{x=1}^{\infty} \frac{1}{x}(1-p)^{x-1}p = \frac{p\ln(p)}{p-1}$

To see this, start with the geometric series $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$.

Integrate both sides, and do some algebra.

Or, use Mathematica or Wolfram Alpha to evaluate the sum.

4. Let $X \sim \text{Unif}[0,1]$. Compute the nth moment of X in two different ways.

(a) Use the formula $E(X^n) = \int_0^1 x^n\,dx$.

$$E(X^n) = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_{x=0}^{x=1} = \frac{1}{n+1}$$

(b) Use the moment generating function $M_X(t)$.

$$M_X(t) = \begin{cases} \frac{e^t - 1}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

We could differentiate $M_X(t)$:

For $t \neq 0$, we find $M_X'(t) = \frac{te^t - e^t + 1}{t^2}$

Due to the t^2 in the denominator, we have to take a limit to evaluate $M_X'(0)$. We could do this:

$$\underbrace{\lim_{t\to 0} \frac{te^t - e^t + 1}{t^2}}_{\text{this limit is } \frac{0}{0}, \text{ which is indeterminate}} = \lim_{t \to 0} \frac{(e^t + te^t) - e^t}{2t} = \lim_{t\to 0}\frac{te^t}{2t} = \lim_{t \to 0} \frac{e^t}{2} = \frac{1}{2}$$

by L'Hospital's rule (differentiate the numerator and the denominator)

This is tedious. Is there a different approach? Yes.

Recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots$

Thus, as a power series,

$$M_X(t) = \frac{1}{t}\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} - 1\right) = \frac{1}{t}\left(t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots\right) = 1 + \frac{t}{2} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$$

$E(X^n)$ is the coefficient of $\frac{t^n}{n!}$ in the power series.

Since $M_X(t) = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{t^n}{n!}$, we see that $E(X^n) = \frac{1}{n+1}$.

BONUS:

(a) Give an example of a random variable X such that $E(X)$ is undefined. (i.e., $E(X)$ diverges to ∞.)

One example is X with pdf $f(x) = \frac{1}{2x^{3/2}}$ for $x \geq 1$.

Expected value:

$$E(X) = \int_1^\infty x \cdot \frac{1}{2x^{3/2}}\,dx = \int_1^\infty \frac{1}{2x^{1/2}}\,dx = \lim_{b\to\infty}\left[x^{1/2}\right]_1^b = \lim_{b\to\infty}\left(\sqrt{b} - 1\right),$$

which diverges to ∞.

Another example is the Cauchy distribution (look it up).

(b) Give an example of a random variable X such that $E(X) < \infty$ and $E(X^2)$ is undefined. (i.e., $E(X^2)$ diverges to ∞.)

One example is X with pdf $f(x) = \frac{3}{2x^{5/2}}$ for $x > 1$.

$$E(X) = \int_1^\infty x \cdot \frac{3}{2x^{5/2}}\,dx = \int_1^\infty \frac{3}{2x^{3/2}}\,dx = \left[\frac{-3}{x^{1/2}}\right]_1^\infty = 0 - (-3) = 3$$

$$E(X^2) = \int_1^\infty x^2 \cdot \frac{3}{2x^{5/2}}\,dx = \int_1^\infty \frac{3}{2x^{1/2}}\,dx = \left[3x^{1/2}\right]_1^\infty,$$ which diverges to ∞.