Worksheet Solutions
Math 262 • 30 October 2023

1. Suppose that emails arrive in your inbox according to a Poisson process with rate 2 emails per hour. Then the time between successive emails is an exponential random variable with mean 30 minutes.

(a) What is the probability that an email arrives in the next 20 minutes?

$X \sim \text{Exp}(2)$ — rate $\lambda = 2$ emails/hour

$P(X < \tfrac{1}{3}) = \int_0^{1/3} 2e^{-2x}\,dx = -e^{-2x}\Big|_0^{1/3} = -e^{-2/3} + 1 \approx 0.4866$

$\tfrac{1}{3}$ hour $= 20$ minutes

(b) What is the probability that you don't receive any emails in the next hour?

We can use the exponential distribution:

$X \sim \text{Exp}(\lambda = 2)$, so $P(X > 1) = \int_1^{\infty} 2e^{-2x}\,dx = -e^{-2x}\Big|_1^{\infty} = e^{-2} \approx 0.135$

$E(X) = \frac{1}{\lambda} = \frac{1}{2}$ hour $= 30$ minutes

Alternatively, use the Poisson distribution:

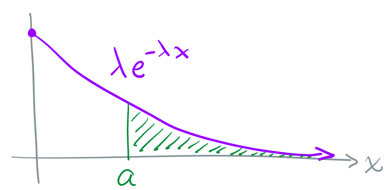
$Y \sim \text{Poisson}(\mu = 2)$, so $P(Y = 0) = e^{-2}\frac{2^0}{0!} = e^{-2}$

recall Poisson pmf $e^{-\mu}\frac{\mu^x}{x!}$

(c) What is the standard deviation of the time until the next email?

$\sigma_X = E(X) = \frac{1}{2}$ hour $= 30$ minutes

2. Let $X \sim \text{Exp}(\lambda)$ and $0 < a < b$.



(a) What is $P(X \geq a)$?

Use the pdf: $P(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_a^{\infty} = 0 - (-e^{-\lambda a}) = \boxed{e^{-\lambda a}}$

exponential tail probability

Or use the cdf: $P(X \leq a) = \int_0^{a} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{a} = -e^{-\lambda a} + e^{0} = 1 - e^{-\lambda a}$

Thus, $P(X \geq a) = 1 - (1 - e^{-\lambda a}) = \boxed{e^{-\lambda a}}$

(b) Show that $\boxed{P(X > b \mid X > a) = P(X > b - a).}$ ← Memoryless Property!

$P(X > b \mid X > a) = \frac{P(X > b \text{ and } X > a)}{P(X > a)} = \frac{P(X > b)}{P(X > a)} = \frac{e^{-\lambda b}}{e^{-\lambda a}}$

definition of conditional probability

$= e^{-\lambda(b-a)} = P(X > b - a)$

(c) What other distribution satisfies the equality in (b)?

the geometric distribution!

(d) The property in (b) is special, in the sense that it doesn't hold for most random variables. For example, if $U \sim$ Unif[0,10], show that $P(U > 4 \mid U > 3) \neq P(U > 1)$.

$$P(U>4 \mid U>3) = \frac{P(U>4)}{P(U>3)} = \frac{\frac{6}{10}}{\frac{7}{10}} = \frac{6}{7} \neq P(U>1) = \frac{9}{10}$$

$$P(U>b \mid U>a) \neq P(U>b-a)$$

3. What is the moment generating function of an exponential random variable?

$X \sim Exp(\lambda)$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_{x=0}^{x=\infty}$$

$$= \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda-t}$$

Note that $\lim_{x\to\infty} \frac{1}{t-\lambda} e^{(t-\lambda)x} = 0$ if $t-\lambda<0$.

$$\boxed{M_X(t) = \frac{\lambda}{\lambda-t} \quad \text{for} \quad t<\lambda}$$

4. Let $X \sim$ Exp(1). Find a formula for $E(X^n)$ for positive integers n.

$M_X(t) = \frac{1}{(1-t)} = (1-t)^{-1}$

$M_X'(t) = (1-t)^{-2}$ $\qquad E(X) = 1$

$M_X''(t) = 2(1-t)^{-3}$ $\qquad E(X^2) = 2$ $\qquad E(X^n) = n!$

$M_X'''(t) = 6(1-t)^{-4}$ $\qquad E(X^3) = 6$

These moments could also be computed by integrals:

$$E(X^n) = \int_0^\infty x^n e^{-x}\,dx$$

The integral above is related to the gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

The gamma function is important for our next distribution: the gamma distribution.

BONUS: For a positive continuous random variable X with pdf f and cdf F, the *hazard rate* (or *failure rate*) is defined by $h(t) = \frac{f(t)}{1-F(t)}$.

(a) Interpret the hazard rate as a conditional probability? *Hint*: $P(t < X < t + \Delta t) \approx f(t)\Delta t$

$$P(t < X < t + \Delta t \mid X > t) = \frac{P(t < X < t + \Delta t)}{P(X > t)} \approx \frac{f(t)\,\Delta t}{1 - F(t)}$$

If X is the lifetime of some item, then $h(t)$ is the rate of failure for items at age t.

(b) Compute the hazard rate for $X \sim \mathrm{Exp}(\lambda)$?

If $X \sim \mathrm{Exp}(\lambda)$, then $h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$.

Thus, the failure rate of an exponential random variable is constant. This is a consequence of the memoryless property.