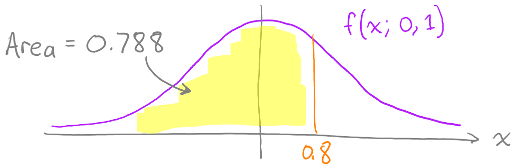
1. Let $Z$ be a standard normal random variable.

(a) What is $P(Z \leq 0.8)$?




As an integral:

$$P(Z \leq 0.8) = \int_{-\infty}^{0.8} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \approx 0.788$$

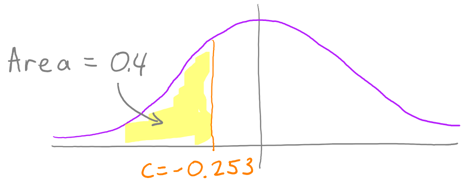
in R: pnorm(0.8, 0, 1) = 0.788

or: pnorm(0.8)

in Mathematica: In[1]:= `CDF[NormalDistribution[], 0.8]`

Out[1]= `0.788145`

(b) What number $c$ is such that $P(Z \leq c) = 0.4$?




This is an inverse CDF calculation that we can only do approximately.

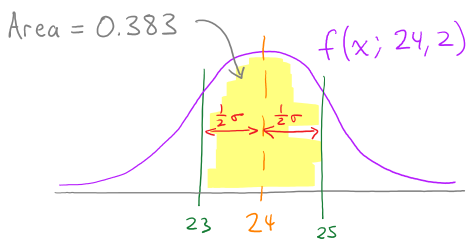
R: qnorm(0.4) = −0.253

Mathematica In[1]:= `InverseCDF[NormalDistribution[], 0.4]`

Out[1]= `-0.253347`

2. Let $X$ be a normal random variable with mean 24 and standard deviation 2.

(a) What is $P(23 \leq X \leq 25)$?




one option in R:

pnorm(25, 24, 2) − pnorm(23, 24, 2)

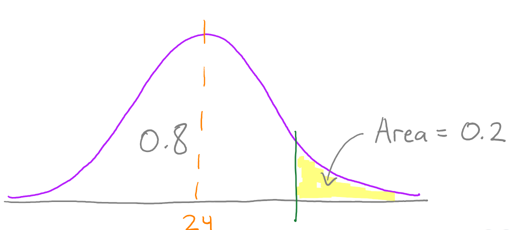
= 0.383

one option in Mathematica:

In[3]:= `Probability[23 ≤ x ≤ 25, x ≈ NormalDistribution[24, 2]] // N`

Out[3]= `0.382925`

(b) What number $c$ is such that $P(X \geq c) = 0.2$?




one option in R:

qnorm(0.8, 24, 2) = 25.68 = c

1−0.2

one option in Mathematica:

In[5]:= `InverseCDF[NormalDistribution[24, 2], 0.8]`

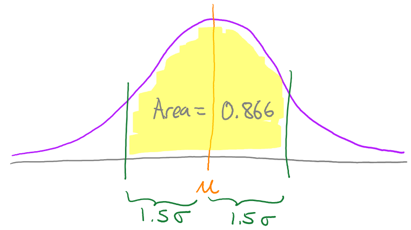
Out[5]= `25.6832`

3. What is the probability that a normal random variable is within 1.5 standard deviations of its mean?

The answer to this question does not depend on the mean or standard deviation of the normal random variable!

We can normalize and compute using $Z \sim N(0,1)$

$$P(\mu - 1.5\sigma < X < \mu + 1.5\sigma) = P(-1.5 < Z < 1.5) = 0.866$$




**R:** pnorm(1.5) − pnorm(−1.5)

**Mathematica:**

```
In[6]:= Probability[-1.5 < x < 1.5, x ≈ NormalDistribution[]]

Out[6]= 0.866386
```

4. Suppose that a fair, 6-sided die is rolled 1000 times. Use a normal distribution to approximate the probability that the number 6 appears between 150 and 200 times (inclusive).

Let $X \sim \text{Bin}\left(1000, \frac{1}{6}\right)$ be the number of 6s rolled.

Then $E(X) = \frac{1000}{6}$ and $\sigma(X) = \sqrt{\frac{5000}{36}} \approx 11.785$.

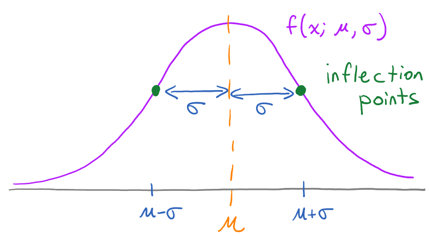
Then $X$ is approximately $Z \sim N\left(\frac{1000}{6}, 11.8\right)$.

$$\underbrace{P(150 \le X \le 200)}_{\text{Binomial distribution gives } 0.92645} \approx P(150 \le Z \le 200) = 0.919$$

[Normal approximation to the binomial distribution is "good" when $np \ge 10$ and $n(1-p) \ge 10$]

**CONTINUITY CORRECTION:** $P(149.5 \le Z \le 200.5) = 0.925$

**BONUS:** Let $f(x)$ denote the pdf of the $N(\mu, \sigma)$ distribution. Show that the points of inflection lie at $x = \mu \pm \sigma$. (*Hint*: differentiate twice with respect to $x$.)




$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{-(x-\mu)}{\sigma^2} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} = \frac{(x-\mu)^2 - \sigma^2}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$f''(x) = 0 \Rightarrow \frac{1}{\sigma^3\sqrt{2\pi}} = \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} \Rightarrow \sigma^2 = (x-\mu)^2 \Rightarrow x - \mu = \pm\sigma \Rightarrow x = \mu \pm \sigma$$

Furthermore:

| | | $\mu-\sigma$ | | $\mu+\sigma$ | |
|---|---|---|---|---|---|
| $f''$ | + | | − | | + |
| $f$ | concave up | | concave down | | concave up |

5. Suppose that emails arrive in your inbox according to a Poisson process with rate 2 emails per hour. Then the time between successive emails is an exponential random variable with mean 30 minutes.

(a) What is the probability that an email arrives in the next 20 minutes?

$X \sim \text{Exp}(2)$ ← rate $\lambda = 2$ emails/hour

$P(X < \frac{1}{3}) = \int_0^{\frac{1}{3}} 2e^{-2x}\,dx = -e^{-2x}\Big|_0^{\frac{1}{3}} = -e^{-\frac{2}{3}} + 1 \approx 0.4866$

$\frac{1}{3}$ hour $= 20$ min

(b) What is the probability that you don't receive any emails in the next hour?

We can use the exponential distribution:

$X \sim \text{Exp}(\lambda = 2)$, so $P(X > 1) = \int_1^{\infty} 2e^{-2x}\,dx = -e^{-2x}\Big|_1^{\infty} = e^{-2}$

$E(X) = \frac{1}{\lambda} = \frac{1}{2}$ hour $= 30$ minutes

Alternatively, use the Poisson distribution:

$Y \sim \text{Poisson}(\mu = 2)$, so $P(Y = 0) = e^{-2}\frac{2^0}{0!} = e^{-2}$

recall Poisson pmf $e^{-\mu}\frac{\mu^x}{x!}$

(c) What is the standard deviation of the time until the next email?

$\sigma_X = E(X) = \frac{1}{2}$ hour $= 30$ minutes

6. Let $X \sim \text{Exp}(\lambda)$ and $0 < a < b$.

(a) What is $P(X \geq a)$?

We will come back to this on Thursday!

(b) Show that $P(X > b \mid X > a) = P(X > b - a)$.

(c) What other distribution satisfies the equality in (b)?

(d) The property in (b) is special, in the sense that it doesn't hold for most random variables. For example, if $U \sim \text{Unif}[0,10]$, show that $P(U > 4 \mid U > 3) \neq P(U > 1)$.