FROM LAST TIME:

7. A sequence of n independent trials are to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that…

(a) …all trials result in successes?

Let A_i be the event that trial i results in success.

By independence, $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n) = \underbrace{p \cdot p \cdots p}_{n \text{ factors}} = p^n$

(b) …at least one trials results in a success?

Probability of no successes: $P(A_1' \cap A_2' \cap \cdots \cap A_n') = (1-p)^n$

Probability of at least one success: $1 - P(A_1' \cap \cdots \cap A_n') = 1 - (1-p)^n$

(c) …exactly k trials result in successes?

We will see this again in Chapter 2.

Any particular sequence of k successes and $n-k$ failures occurs with probability $p^k(1-p)^{n-k}$. There are $\binom{n}{k}$ such sequences.

Thus, $P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}$.

This factor is important!

8. If A and B are independent events with positive probability, show that they cannot be mutually exclusive.

Given: $P(A) > 0$, $P(B) > 0$, and $P(A \cap B) = P(A)P(B)$.

Thus $P(A \cap B) > 0$, so the events can occur simultaneously, meaning they are not mutually exclusive.

TODAY'S WORKSHEET:

1. Consider an urn containing four balls, numbered 110, 101, 011, and 000. One ball is drawn at random. For $k = 1,2,3$, let A_k be the event that the k^{th} digit is a 1 on the ball that is drawn.

(a) Are the events A_1, A_2, and A_3 pairwise independent? Why or why not?

Yes: $P(A_i) = \frac{1}{2}$ for any $i \in \{1,2,3\}$

$P(A_i \cap A_j) = \frac{1}{4}$ for any $i, j \in \{1,2,3\}$

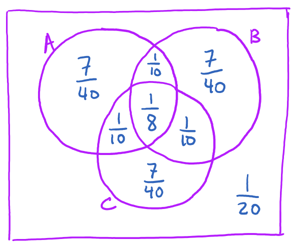
(b) Are the events A_1, A_2, and A_3 mutually independent? Why or why not?

No: $P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3)$

2. Create an example of three events A, B, C such that $P(A \cap B \cap C) = P(A)P(B)P(C)$ but the events are not mutually independent.

This is tricky! Trial-and-error is a fine approach.

Here is one example:



$P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$

However:

$P(A \cap B) = \frac{9}{40} \neq P(A)P(B)$

(and similarly for $A \cap C$ and $B \cap C$)

A, B, and C are not pairwise independent, and thus not mutually independent.

SIMULATION SOLUTIONS

There are lots of ways to implement the same simulation, so your solutions may differ from these.

3. Suppose you flip two unfair coins, one of which lands heads with probability 0.4, and the other lands heads with probability 0.6. Estimate the probability that both land heads.

R:

```
count <- 0              # count starts at zero
for(i in 1:10000){      # loop 10000 times
  # generate random numbers between 0 and 1
  r <- runif(1)
  s <- runif(1)

  # if both coins are heads, then increment counter
  if(r < 0.4 && s < 0.6){
    count <- count + 1
  }
}
print(count) # this is the number of times both coins land heads
```

```
# more concise code to solve the same problem as above
r <- runif(10000) # 10000 flips of the first coin
s <- runif(10000) # 10000 flips of the second coin
count = (r < 0.4) & (s < 0.6)
#print(count)
print(sum(count))
```

Mathematica:

```
count = 0;
Do[
 r = RandomReal[1];
 s = RandomReal[1];
 If[r < 0.4 && s < 0.6, count += 1],
 10 000]
count
```

Exact probability: $(0.4)(0.6) = 0.24$

4. Use simulation to approximate the probability that at least two sixes appear in three rolls of standard, fair dice.

R:

```
# simulate the probability that at least two sixes appears in
#   three rolls of a standard, fair die
c <- 0
for(i in 1:10000){
  dice <- sample(1:6, 3, TRUE) # three die rolls
  sixes <- sum(dice == 1)        # number of ones in the rolls
  if(sixes >= 2){
    c <- c + 1    # increment counter
  }
}
print(c/10000)
```

Mathematica:

```
count = 0;
Do[
 dice = RandomChoice[Range[6], 3];
 sixes = Count[dice, 6];
 If[sixes ≥ 2, count += 1],
 10 000]
N[count / 10 000]
```

Exact Probability:

$$P(\text{at least two sixes}) = P(\text{exactly two sixes}) + P(\text{all three sixes})$$

$$= 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^3 = \frac{16}{216} \approx 0.074$$

5. Suppose there are 3000 students at St. Olaf College. Estimate the probability that at least 18 students share the same birthday.

R:

```
# simulate the probability that at least 18 people out of 3000 people
#   have the same birthday
count <- 0
for(i in 1:10000){
  bdays <- sample(1:365, 3000, TRUE)
  tab <- table(bdays)
  if(max(tab) >= 18){
    count <- count + 1
  }
}
print(count/10000)
```

Mathematica:

```
count = 0;
Do[
 bdays = RandomChoice[Range[365], 3000];
 tab = BinCounts[bdays];
 If[Max[tab] ≥ 18, count += 1],
 10000]
N[count / 10000]
```

The exact probability is tough to determine, but it's about 0.54.