18 March 2024

The Logistic Map

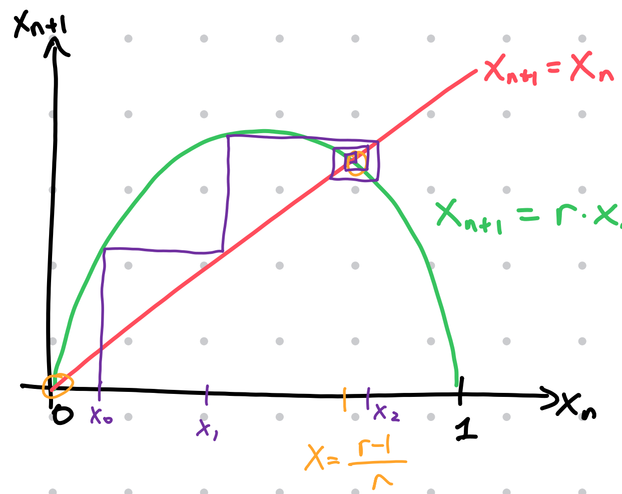
DISCUSS:

What did you observe for $r > 3$?

$r \approx 3.449$?
3.450 ?

At what value(s) does a bifurcation occur?

How would you describe the bifurcation(s)?



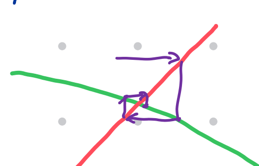
If $0 < r < 1$, then only one fixed point at $x = 0$, and all trajectories converge to this.

If $1 < r < 3$, then there is another fixed point at $x = \frac{r-1}{r}$, and all trajectories converge to this. (Trajectories now move away from $x = 0$.)

If $1 < r < 3$, then slope of the parabola at the fixed point is between -1 and 1.

$y = rx(1-x)$

$y = -rx^2 + rx$

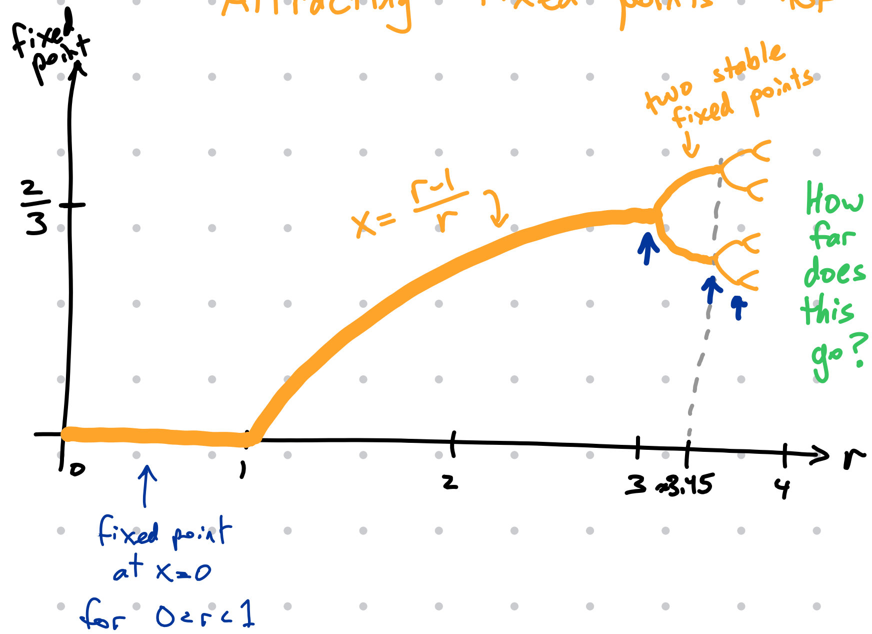


This causes the trajectory to converge to the fixed point.

$\left.\begin{array}{l} \frac{dy}{dx} = -2rx + r \\ \text{fixed point: } x = \frac{r-1}{r} \end{array}\right\} -2\cancel{r}\left(\frac{r-1}{\cancel{r}}\right) + r = 2 - 2r + r = 2 - r$

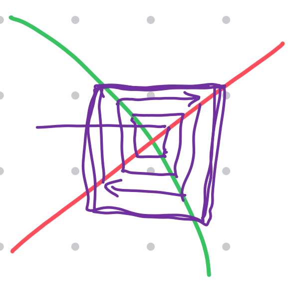
Period-Doubling Bifurcations

Attracting fixed points for $0 < r < 4$



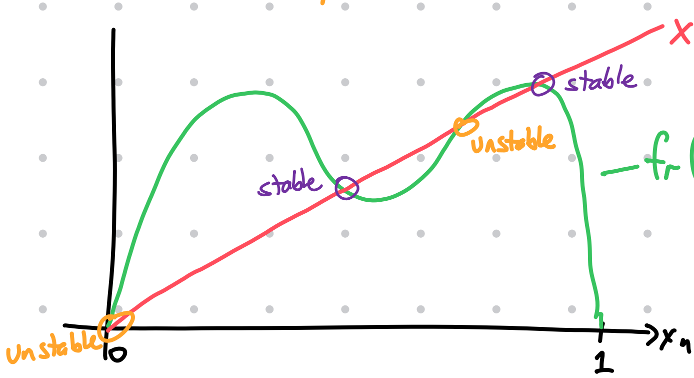
If $r = 3$, then fixed point is $\frac{3-1}{3} = \frac{2}{3}$

If r is somewhat larger than 3: the trajectory spirals outward from the fixed point, before entering oscillation between two values.



The fixed point at $x = \frac{r-1}{r}$ is still present, but now unstable.

Oscillation between two values suggests fixed points of $\underbrace{f_r(f_r(x))}_{\text{compose } f_r \text{ with itself}}$



$$f_r(f_r(x)) = f_r(r \cdot x(1-x))$$
$$= r \cdot (rx(1-x)) \cdot (1 - (rx(1-x)))$$