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$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ \dots$

Could the Fibonacci numbers satisfy

$$F_{3n} = a F_n^3 + b F_n^2 + c F_n$$

for some constants a, b, c and all indexes $n \geq 0$?

Idea: choose 3 values of n, and write 3 equations.
Solve for a, b, c.
See whether the identity holds for all $n \geq 0$.

For odd n: $a = 5,\ b = 0,\ c = -3$

So: $F_{3n} = 5F_n^3 - 3F_n$

CONJECTURE: The Fibonacci numbers satisfy

Fibonacci 3n-Identity

$$F_{3n} = 5F_n^3 + (-1)^n 3F_n$$

for all indexes $n \geq 0$.

Linear System:

$$\begin{bmatrix} 2 \\ 34 \\ 610 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 125 & 25 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

↑ Vector ↑ Matrix

$$\begin{cases} 2 = 1a + 1b + 1c & (n=1) \\ 34 = 8a + 4b + 2c & (n=3) \\ 610 = 125a + 25b + 5c & (n=5) \end{cases}$$