MATH 242 – 1 Oct. 2025

Change the Fibonacci recurrence to something else

$$\underbrace{G_0 = a, \quad G_1 = b,}_{\text{starting values}} \quad \underbrace{G_n = \boxed{r} \cdot G_{n-1} + \boxed{s}\, G_{n-2}}_{\text{recurrence}} \quad \text{for } n > 1$$

Pell Sequence:

def: $P_0 = 0, \quad P_1 = 1, \quad P_n = \boxed{2}\, P_{n-1} + P_{n-2}$

$0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots$

Application: approximating $\sqrt{2} \approx 1.414213\ldots$

The equation $x^2 - 2y^2 = 0$ $\qquad x^2 = 2y^2$

$$\frac{x^2}{y^2} = 2$$

If $\sqrt{2} = \frac{x}{y}$, then we would have an integer solution to $x^2 - 2y^2 = 0$.

But $x^2 - 2y^2 = 0$ has no integer solutions.

What about $x^2 - 2y^2 = \pm 1$?

integer solutions

$1^2 - 2(1)^2 = -1$

$3^2 - 2(2)^2 = 1$

$7^2 - 2(5)^2 = -1$

$\frac{x}{y}$: $\frac{1}{1} = 1, \quad \frac{3}{2} = 1.5, \quad \frac{7}{5} = 1.4$

$\frac{17}{12}, \quad \frac{41}{29}, \quad \ldots$

denominators: Pell sequence

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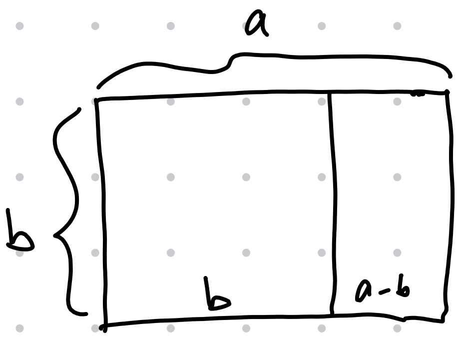
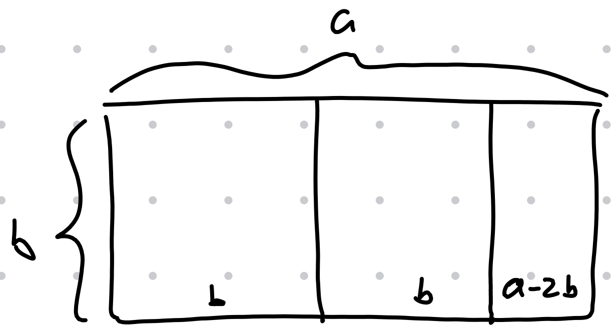
$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 2.41423\ldots = 1+\sqrt{2}$$

SILVER RATIO



$$\phi = \frac{a}{b} = \frac{1+\sqrt{5}}{2} = \frac{b}{a-b}$$

Golden Ratio



$$\frac{a}{b} = \frac{b}{a-2b} = 1+\sqrt{2}$$

Silver Ratio