Big-O notation

"$f(n)$ is $O(g(n))$" means

- $f(n)$ is asymptotically similar to <u>or</u> smaller than $g(n)$
- $f(n)$ does not grow faster than a constant multiple of $g(n)$
- There exist constants c and k such that

$$|f(n)| \le c \cdot |g(n)| \quad \text{whenever} \quad n > k.$$

EXAMPLES:

- $f(x) = x + 1$ is $O(x)$
- $f(x) = 1000x$ is $O(x)$
- $f(x) = 5x^3 + 17x - 3$ is $O(x^3)$ ... and $O(x^4)$
- $f(x) = 2^x$ is not $O(x^2)$
- $f(x) = 2^x + x^{17}$ is $O(2^x)$

A relation R on a set A is a subset of $A \times A$.

Math 234
Relations

$R \subseteq A \times A$. We write $x R y$ if $(x,y) \in R$.

Day 22

1. Let $A = \{n \in \mathbf{Z} \mid -20 \le n \le 20\}$ and define relation R on A by $R = \{(n_1, n_2) \mid n_1^2 = n_2\}$.

   (a) Is it true that $3\ R\ 9$?

   Yes, since $3^2 = 9$, $(3,9) \in R$.

   but $(9,3) \notin R$, so $\cancel{9 R 3}$

   (b) Is it true that $-4\ R\ 16$?

   Yes, since $(-4)^2 = 16$, $(-4,16) \in R$

   (c) Is it true that $5\ R\ 10$?

   No, since $5^2 \ne 10$, $(5,10) \notin R$

   (d) Write out every element in the set R.

   $R = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), (-3,9), (4,16), (-4,16)\}$

   (e) Write out every element in the set R^{-1}, the inverse relation of R.

   $R^{-1} = \{(0,0), (1,1), (1,-1), (4,2), (4,-2), (9,3), (9,-3), (16,4), (16,-4)\}$

   (f) Is the relation R reflexive? Is it symmetric? Is it transitive?

   Not reflexive since $(2,2) \notin R$.
   Not symmetric since $(2,4) \in R$ but $(4,2) \notin R$.
   Not transitive since $(2,4) \in R$, $(4,16) \in R$, but $(2,16) \notin R$.

2. For the same set A as above, let $S = \big\{(n_1, n_2) \bigm| |n_1| \le |n_2|\big\}$.

   (a) Is it true that $-3\ S\ 9$?

   Yes: $|-3| \le |9|$

   (b) Is it true that $9\ S\ 3$?

   No: $|9| \nleq |3|$

   (c) Is it true that $3\ S\ -9$?

   Yes: $|3| \le |-9|$

   (d) Is it true that $10\ S^{-1}\ -7$?

   true iff $-7\ S\ 10$, which is true since $|7| \le |10|$

   (e) Is the relation S reflexive? Is it symmetric? Is it transitive?

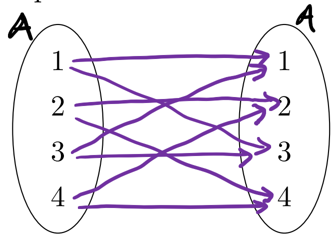
   Reflexive: since $(x,x) \in S\ \forall x \in A$ $\quad |x| \le |x|$

   Symmetric: No, since $(1,2) \in S$ but $(2,1) \notin S$

   Transitive: Yes, if $|x| \le |y|$ and $|y| \le |z|$, then $|x| \le |z|$

3. Let $A = \{1,2,3,4\}$, and define relation $R = \{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}$.

(a) Complete the arrow diagram to depict relation R.



(b) Is R reflexive? Is it symmetric? Is it transitive?

Yes reflexive: $(1,1), (2,2), (3,3), (4,4) \in R$

Yes symmetric: if $(x,y) \in R$, then $(y,x) \in R$

Yes transitive: if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

(c) Draw an arrow diagram to depict relation R^{-1}.

Same as for R

4. Define a relation Q on $\mathbf{R}$ as follows: For all real numbers x and y, $x\ Q\ y \Leftrightarrow x - y$ is rational. Is Q reflexive? Is it symmetric? Is it transitive?

$(x,y) \in Q$ if and only if $x-y$ is rational

Yes reflexive:
$\forall x \in \mathbb{R}$, $x - x = 0$ is rational, so $x Q x$.

Yes symmetric:
$\forall x,y \in \mathbb{R}$, if $x-y$ is rational, then $y-x$ is also rational
$x-y = \frac{p}{q}$ $\qquad y-x = \frac{-p}{q}$

Thus if $(x,y) \in Q$, so is (y,x).

Yes transitive: if $x-y$ is rational and $y-z$ is rational, then $x-z = (x-y)+(y-z)$ is also rational.

We did not do this page in class.

5. Let X be a finite set. Define the following relations on $\mathscr{P}(X)$, the power set of X. Is each relation reflexive? Symmetric? Transitive?

(a) For all $A, B \in \mathscr{P}(X)$, $A\ \mathsf{E}\ B \Leftrightarrow$ the number of elements in A equals the number of elements in B.

Reflexive: Yes, since $N(A) = N(A)$ for any $A \in \mathscr{P}(X)$.

Symmetric: Yes. If $N(A) = N(B)$, then $N(B) = N(A)$.

Transitive: Yes. If $N(A) = N(B)$ and $N(B) = N(C)$, then $N(A) = N(C)$.

(b) For all $A, B \in \mathscr{P}(X)$, $A\ \mathsf{L}\ B \Leftrightarrow$ the number of elements in A is less than the number of elements in B.

Not reflexive, since $N(A) \not< N(A)$.

Not symmetric, since if $N(A) < N(B)$, then $N(B) \not< N(A)$.

Transitive: Yes, since if $N(A) < N(B)$ and $N(B) < N(C)$, then $N(A) < N(C)$.

(c) For all $A, B \in \mathscr{P}(X)$, $A\ \mathsf{N}\ B \Leftrightarrow$ the number of elements in A is not equal to the number of elements in B.

Not reflexive: Since $N(A) = N(A)$, it's <u>not</u> true that $A\ N\ B$.

Symmetric: Yes, since if $N(A) \neq N(B)$, then $N(B) \neq N(A)$.

Not transitive: It could be true that $N(A) \neq N(B)$ and $N(B) \neq N(C)$ but $N(A) = N(C)$.

$R \subseteq A \times A$ If $(a,b) \in R$, then $\underbrace{a \, R \, b}$.

↳ a is related to b by R

6. Suppose R and S are reflexive relations on the same set A.

(a) Is $R \cup S$ a reflexive relation on A? Prove your answer is correct.

R reflexive means $(a,a) \in R \ \forall \ a \in A$

S reflexive means $(a,a) \in S \ \forall \ a \in A$.

note the conclusion would follow from either of these individually

Thus, for any $a \in A$, $(a,a) \in R \cup S$, so $R \cup S$ is reflexive.

(b) Is $R \cap S$ a reflexive relation on A? Prove your answer is correct.

R reflexive $\Rightarrow (a,a) \in R \ \forall \ a \in A$

S reflexive $\Rightarrow (a,a) \in S \ \forall \ a \in A$

Both of these statements imply that $(a,a) \in R \cap S \ \forall \ a \in A$, so $R \cap S$ is reflexive.

(c) Is $R - S$ a reflexive relation on A? Prove your answer is correct.

Since $(a,a) \in S$, $(a,a) \notin R-S$ for any $a \in A$.

Thus $R-S$ is not reflexive.