Math 234

Functions, Cardinality, and Infinite Sets — Day 20

1. Fill in the blank:

   Sets A and B have the same cardinality if and only if there exists a function $f: A \to B$ that is $\underbrace{\text{one-to-one and onto}}_{\text{a bijection}}$.

2. Which of the following sets have the same cardinality?

   $A = \{1, 2, 3\}$

   $B = \{2, 4, 6, \ldots, 400\}$

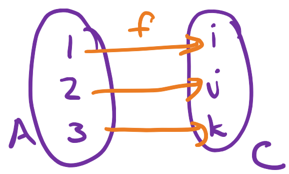
   $C = \{i, j, k\}$

   $M = \{2, 4, 6, \ldots\}$

   $S = \{1, 2, 3, \ldots, 200\}$

   $\mathbf{Z}^+ = \{1, 2, 3, 4, \ldots\}$

   A and C have the same cardinality



   B and S: define $f: B \to S$ via $f(x) = \frac{x}{2}$ is one-to-one and onto

   M and $\mathbb{Z}^+$: define $g: M \to \mathbb{Z}^+$ via $g(x) = \frac{x}{2}$ is one-to-one and onto

3. Which of the sets in #2 are *countable*?

   All of them

   → finite or $\underbrace{\text{countably infinite}}_{\text{same cardinality as } \mathbb{Z}^+}$

4. Prove that the set of all square numbers is countable.

   $S = \{0, 1, 4, 9, 16, 25, 36, \ldots\}$

   $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$

   define $f: \mathbb{Z}^+ \to S$ by $f(n) = (n-1)^2$

5. Prove that **Z** is countable.

$\mathbb{Z} = \{ \ldots,$	$-4,$	$-3,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3,$	$4,$	$\ldots \}$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
Count:	9	7	5	3	1	2	4	6	8	

$\mathbb{Z}^+ = 1, 2, 3, 4, 5, 6, 7, \ldots$

$\mathbb{Z} = 0, 1, -1, 2, -1, 3, -3, \ldots$

6. Let $\mathbf{Q}^+$ be the set of all positive rational numbers. Is $\mathbf{Q}^+$ countable? Why or why not?

counts:

$$\begin{array}{cccccc}
{}^{1}\frac{1}{1} \rightarrow & {}^{2}\frac{1}{2} & {}^{5}\frac{1}{3} \rightarrow & {}^{6}\frac{1}{4} & \frac{1}{5} & \cdots \\
{}^{3}\frac{2}{1} & \cancel{\frac{2}{2}} & {}^{7}\frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
{}^{4}\frac{3}{1} & {}^{8}\frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
{}^{9}\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Yes! We can put all positive rationals into a list by going through the grid at left, one diagonal after another.

7. Is $\mathbf{Z}^+ \times \mathbf{Z}^+$ countable? Why or why not? Yes!

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
(5,1) & \vdots & \vdots & \vdots & \ddots \\
\vdots & & & &
\end{array}$$

Similar to the previous problem, put all pairs (m,n) into a grid, as at left. Then start in the upper left corner and order all pairs, one diagonal after another.

8. Let $S = \{x \in \mathbf{R} \mid 0 < x < 1\}$. Is S countable? Why or why not?

No. Proof by contradiction:

Suppose S is countable. Then the elements of S can be put into a list:

$$\begin{array}{ll}
1: & 0.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\ldots \leftarrow \text{decimal digits} \\
2: & 0.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\ldots \\
3: & 0.\,a_{31}\,a_{32}\,a_{33}\,a_{34}\ldots \\
4: & 0.\,a_{41}\,a_{42}\,a_{43}\,a_{44}\ldots \\
\vdots & \vdots
\end{array}$$

Construct a decimal number $d = 0.\,d_1 d_2 d_3 d_4 \ldots$ (decimal digits)

where $d_n = \begin{cases} 4 & \text{if } a_{nn} = 2 \\ 2 & \text{if } a_{nn} \neq 2 \end{cases}$

Now d is not in the list since it differs from the n^{th} number in digit n, for all n. Contradiction.

9. Let $M = \{x \in \mathbf{R} \mid 0 < x < 0.1\}$. Show that M has the same cardinality as S from the previous problem.

Define $f: S \to M$ by $f(x) = \frac{x}{10}$.

Since f is one-to-one and onto, this shows that M has the same cardinality as S.

10. Suppose A and B are countable sets. Is $A \cup B$ countable?

Yes! Since A is countable, denote the elements of A as $a_1, a_2, a_3, \ldots$. Similarly, denote the elements of B as $b_1, b_2, b_3, \ldots$.

Define $f: \mathbb{Z}^+ \to A \cup B$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$

So $\{f(1), f(2), f(3), f(4), \ldots\} = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}$.

Since f is one-to-one and onto, $A \cup B$ is countable.

11. Is the set of irrational numbers countable? Why or why not?

No. Proof by counterexample:

Let $\mathbb{Q}^c$ denote the irrationals. Suppose $\mathbb{Q}^c$ is countable.

Since $\mathbb{Q}$ is countable, by the previous problem $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ is countable.

But this is a contradiction since $\mathbb{R}$ is uncountable.

Therefore $\mathbb{Q}^c$ is not countable.