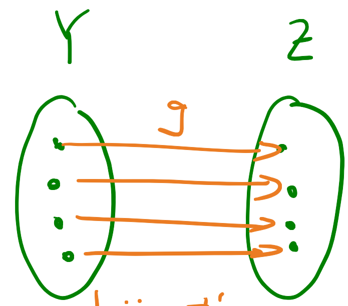
Math 234

Functions: Inverse and Composition Day 19

1. Suppose Y and Z are sets and $g : Y \to Z$ is a one-to-one and onto function. What can be inferred in the following situation?



bijection between Y and Z

(a) s_1 and s_2 are elements of Y and $g(s_1) = g(s_2)$.

s_1 and s_2 are the same

$s_1 = s_2$ since g is one-to-one

(b) $s/2$ and $t/2$ are elements of Y and $g(s/2) = g(t/2)$.

$\frac{s}{2} = \frac{t}{2}$ since g is one-to-one,

so $s = t$

(c) For some function f whose domain includes x_1 and x_2, $f(x_1)$ and $f(x_2)$ are elements of Y and $g(f(x_1)) = g(f(x_2))$.

$f(x_1) = f(x_2)$ since g is one-to-one

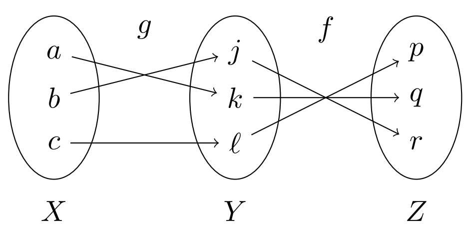
we don't know if x_1 and x_2 are the same

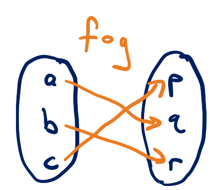
2. Let $X = \{a, b, c\}$, $Y = \{d, e, f\}$ and $Z = \{g, h, i\}$. Define $g : X \to Y$ and $f : Y \to Z$ by the diagram below.

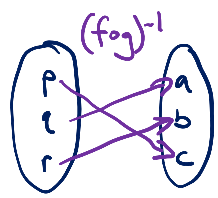
FUNCTION COMPOSITION

$(f \circ g)(x) = f(g(x))$

do g first, then f



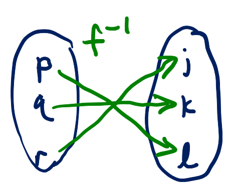


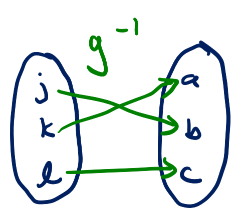


(a) Draw an arrow diagram for the composition $f \circ g$.

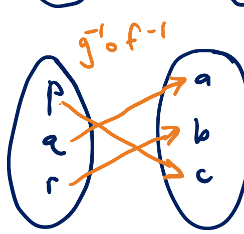
(b) Draw an arrow diagram for the composition $(f \circ g)^{-1}$.

(c) Draw arrow diagrams for g^{-1} and f^{-1}.





(d) Draw an arrow diagram for $g^{-1} \circ f^{-1}$.



(e) How are $(f \circ g)^{-1}$ and $g^{-1} \circ f^{-1}$ related?

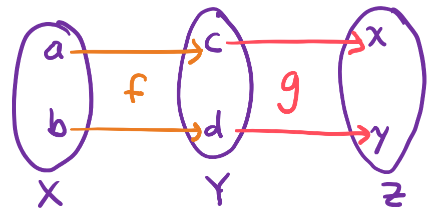
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

3. Suppose $f : X \to Y$ is one-to-one and $g : Y \to Z$ is onto. Consider the composition $g \circ f : X \to Z$.

(a) Is it *possible* that $g \circ f$ is one-to-one? If so, find examples of functions f and g so that $g \circ f$ is one-to-one. If not, explain why not.

Yes.

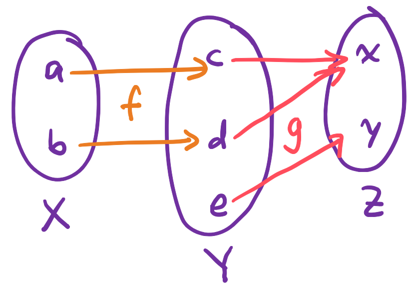
example:



(b) Is it *certain* that $g \circ f$ is one-to-one? If not, find examples of functions f and g so that $g \circ f$ is not one-to-one. If yes, prove that $g \circ f$ is one-to-one.

No

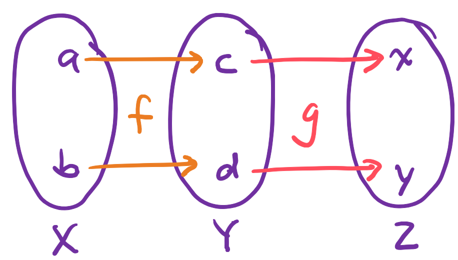
counterexample:



(c) Is it *possible* that $g \circ f$ is onto? If so, find examples of functions f and g so that $g \circ f$ is onto. If not, explain why not.

Yes

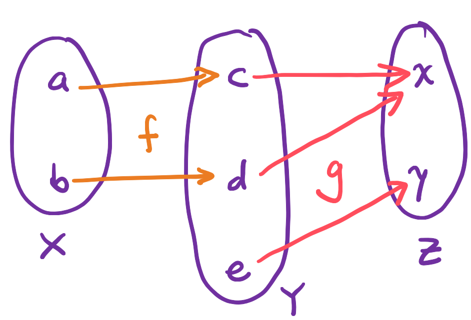
example:



(d) Is it *certain* that $g \circ f$ is onto? If not, find examples of functions f and g so that $g \circ f$ is not onto. If yes, prove that $g \circ f$ is onto.

No.

counterexample:



4. Let $f(x) = \dfrac{x+4}{3x-2}$.

(a) The function definition for f is incomplete because the domain of f is not specified. What is the largest set of real numbers that could be the domain of f?

We simply have to avoid $x = \frac{2}{3}$

Let $T = \mathbb{R} - \{\frac{2}{3}\}$.

Then define $f : T \to \mathbb{R}$ by $f(x) = \dfrac{x+4}{3x-2}$.

(b) Find the inverse function f^{-1}. What is the domain of f^{-1}?

$y = f(x)$ and $x = f^{-1}(y)$

Let $y = \dfrac{x+4}{3x-2}$ and solve for x.

$$y(3x-2) = x+4$$
$$3xy - 2y = x+4$$
$$3xy - x = 4+2y$$
$$x(3y-1) = 4+2y$$

So $x = \dfrac{4+2y}{3y-1}$

Thus $f^{-1}(y) = \dfrac{4+2y}{3y-1}$, with domain $\mathbb{R} - \{\frac{1}{3}\}$

(c) Check your answer from part (b) by computing $f^{-1}(f(x)) = x$ for all x in the domain of f.

$$f^{-1}(f(x)) = f^{-1}\Bigg(\underbrace{\frac{x+4}{3x-2}}_{y}\Bigg) = \frac{\left(4 + 2\frac{x+4}{3x-2}\right)(3x-2)}{\left(3\frac{x+4}{3x-2} - 1\right)(3x-2)} = \frac{4(3x-2) + 2(x+4)}{3(x+4) - 1(3x-2)}$$

$$= \frac{12x - \cancel{8} + 2x + \cancel{8}}{\cancel{3x} + 12 - \cancel{3x} + 2} = \frac{14x}{14} = x$$

$f^{-1}(f(x)) = x$ for all $x \in \mathbb{R}$, $x \neq \frac{2}{3}$

(d) Is f a bijection between two sets of real numbers? If so, which sets? If not, why not?

$T = \mathbb{R} - \{\frac{2}{3}\}$ and let $S = \mathbb{R} - \{\frac{1}{3}\}$

f is a bijection between T and S.

5. Given any set X and any functions $f : X \to X$, $g : X \to X$, and $h : X \to X$. If h is one-to-one and $h \circ f = h \circ g$, is it $\underbrace{\text{true that } f = g?}_{\text{Yes.}}$ Either give a proof or a counterexample.

☜ Problem 21 in Section 7.3

PROOF: Let X be any set and f, g, h functions from X to X such that h is one-to-one and $h \circ f = h \circ g$.

Let $x \in X$.

Since $h \circ f = h \circ g$, we have $(h \circ f)(x) = (h \circ g)(x)$, which means that $h(f(x)) = h(g(x))$.

Since h is one-to-one, this implies that $f(x) = g(x)$.

Therefore, $f = g$.

6. Given any set X and any functions $f : X \to X$, $g : X \to X$, and $h : X \to X$. If h is one-to-one and $f \circ h = g \circ h$, is it $\underbrace{\text{true that } f = g?}_{\text{Yes.}}$ Either give a proof or a counterexample.

☜ Problem 22 in Section 7.3

PROOF: Let X be any set and f, g, h functions from X to X such that h is one-to-one and $f \circ h = g \circ h$.

Since $h : X \to X$ is one-to-one with the same domain and range, h is also onto.

Let $y \in X$. Since h is onto, there exists $x \in X$ such that $h(x) = y$.

Since $f \circ h = g \circ h$, we have $(f \circ h)(x) = (g \circ h)(x)$, meaning that $f(h(x)) = g(h(x))$, which means $f(y) = g(y)$.

Therefore, $f = g$.