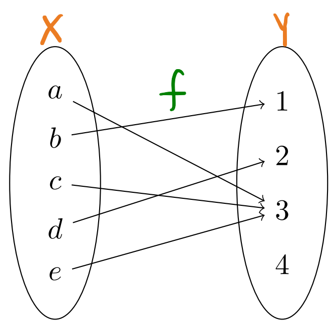
Math 234

Functions | Day 18

1. A student defines a function $g : \mathbf{Q} \to \mathbf{Z}$ by the rule
$$g\left(\frac{m}{n}\right) = m - n$$
for all rational numbers $\frac{m}{n}$. Is this function well defined? Why or why not?

$g\left(\frac{1}{2}\right) = 1-2 = -1$ but $g\left(\frac{2}{4}\right) = 2-4 = -2$ and $\frac{1}{2} = \frac{2}{4}$

2. Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3, 4\}$. Define the function $f : X \to Y$ by the following arrow diagram.



(a) What are the values of $f(a)$, $f(b)$ and $f(c)$?

$f(a) = 3$, $f(b) = 1$, $f(c) = 3$

(b) What is the *domain* of f?

X

(c) What is the *codomain* of f?

$Y = \{1, 2, 3, 4\}$

(d) What is the *range* of f?

$\{1, 2, 3\}$

(e) Is a an *inverse image* of 3?

$f(a) = 3$, so yes.

(f) What is the *inverse image* of 3?

$f^{-1}(3) = \{a, c, e\}$

(g) Is f well defined?

Yes

(h) Is f one-to-one?

No, since $f^{-1}(c)$ has more than one element.

(i) Is f onto?

No, since there is no $x \in X$ such that $f(x) = 4 \in Y$.

Power set of the integers — all possible sets of integers

3. Let $F : \mathscr{P}(\mathbf{Z}) \to \mathbf{R}$ be the function defined by

$$F(A) = \max(A) - \min(A).$$

For example, $F(\{-2, 5, 2\}) = 5 - (-2) = 7$.

(a) What is the value of $F(\{1, 3, 19, -3\})$? $= 19 - (-3) = 22$

(b) What is the *domain* of F?

$\mathcal{P}(\mathbb{Z})$

(c) What is the *codomain* of F?

$\mathbb{R}$

(d) What is the *range* of F?

$\mathbb{Z}^{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$

(e) Is $\{5, -7, 2, 15\}$ an *inverse image* of 20? ...of 22?

no yes, since $15 - (-7) = 22$

(f) What is the *inverse image* of 6?

$\{X \in \mathcal{P}(\mathbb{Z}) \mid f(X) = 6\} = \{X \in \mathcal{P}(\mathbb{Z}) \mid \max(X) - \min(X) = 6\}$

(g) Is F well defined?

No, since $\emptyset \in \mathcal{P}(\mathbb{Z})$ but $F(\emptyset)$ is not defined.

(h) Is F one-to-one?

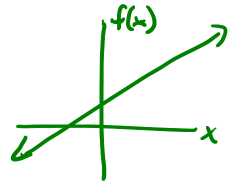
No

(i) Is F onto?

No

4. Draw your own arrow diagram to define a function that is one-to-one but not onto. Then draw your own arrow diagram to define a function that is onto but not one-to-one.

5. Define $f : \mathbf{R} \to \mathbf{R}$ by the rule $f(x) = 5x + 3$.



(a) Is f one-to-one? Prove that your answer is correct.

IDEA: Suppose that x, y in the domain are such that $f(x) = f(y)$. Show that in fact $x = y$.

PROOF: Suppose $x, y \in \mathbb{R}$ are such that $f(x) = f(y)$. That means $5x+3 = 5y+3$. Then $5x = 5y$, and so $x = y$. Thus, f is one-to-one.

(b) Is f onto? Prove that your answer is correct.

IDEA: Suppose that y is in the codomain. Show that there is some x in the domain such that $y = f(x)$.

PROOF: Suppose $y \in \mathbb{R}$.
Let $x = \frac{y-3}{5}$. Then $f(x) = f\left(\frac{y-3}{5}\right) = 5\left(\frac{y-3}{5}\right) + 3$
So $f(x) = y$.
Thus, f is onto.

scratch work:
$y = f(x)$
$y = 5x+3$
$y-3 = 5x$
$\frac{y-3}{5} = x$

6. Define $g : \mathbf{Z} \to \mathbf{Z}$ by the rule $g(n) = 3n + 1 \pmod 7$.

$g(4) = 3(4)+1 = 13 \bmod 7$
$\equiv 6$
so $g(4) = 6$

(a) Is g one-to-one? Prove that your answer is correct.

No. Consider $n = 0$ and $m = 7$:

$g(0) = 1$

$g(7) = 3(7) + 1 \equiv 1 \pmod 7$

So $g(0) = g(7)$ and g is not one-to-one.

(b) Is g onto? Prove that your answer is correct.

No. There is no $n \in \mathbb{Z}$ such that $g(n) = 8$.

The **floor function** assigns to each $x \in \mathbf{R}$ the largest integer that is less than or equal to x. The value of the floor function at x is denoted $\lfloor x \rfloor$. round down

The **ceiling function** assigns to each $x \in \mathbf{R}$ the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted $\lceil x \rceil$. round up

7. Compute the following:

$$\left\lfloor \frac{1}{2} \right\rfloor = 0 \qquad \left\lceil \frac{5}{2} \right\rceil = 3 \qquad \lfloor 3.2 \rfloor = 3 \qquad \lceil 7 \rceil = 7 \qquad \lfloor -3 \rfloor = -3$$

8. Prove or disprove the following statements about the floor and ceiling functions.

(a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x. True!

Let $x \in \mathbb{R}$. Then let $n = \lceil x \rceil$, which is an integer, so $\lfloor \lceil x \rceil \rfloor = n$.

Thus $\lfloor \lceil x \rceil \rfloor = n = \lceil x \rceil$.

(b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y. False!

Counterexample: $x = y = \frac{1}{2}$

$\lfloor x + y \rfloor = \lfloor \frac{1}{2} + \frac{1}{2} \rfloor = 1$

$\lfloor x \rfloor + \lfloor y \rfloor = \lfloor \frac{1}{2} \rfloor + \lfloor \frac{1}{2} \rfloor = 0 + 0 = 0$

(c) $\left\lceil \frac{\lceil x/2 \rceil}{2} \right\rceil = \left\lceil \frac{x}{4} \right\rceil$ for all real numbers x.

True! For the proof, consider cases modulo 4.

(d) $\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.

False: Counterexample: let $x = \frac{1}{2}$, then

$\left\lfloor \sqrt{\lceil \frac{1}{2} \rceil} \right\rfloor = \lfloor \sqrt{1} \rfloor = \lfloor 1 \rfloor = 1$

$\left\lfloor \sqrt{\frac{1}{2}} \right\rfloor = 0$ since $0 < \sqrt{\frac{1}{2}} < 1$