Math 234

Permutations and Binomial Coefficients — Day 17

1. How many different ways can the letters in the following words be arranged?

   (a) BOOKKEEPER — 10 letters total

   👈 triple-double letters!

$$\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{10!}{\cancel{7!}\,3!}\cdot\frac{\cancel{7!}}{\cancel{5!}\,2!}\cdot\frac{\cancel{5!}}{\cancel{3!}\,2!}\cdot\frac{\cancel{3!}}{\cancel{2!}\,1!}\cdot\frac{\cancel{2!}}{1!\,1!}\cdot\frac{1!}{1!\,1!}$$

E O K B P R

10 letters total →

$$\frac{10!}{3!\,2!\,2!} = \frac{10!}{3!\,2!\,2!} = 151{,}200$$

E → 3!, O, K

   (b) UNSUCCESSFULLY

$$\frac{14!}{3!\,3!\,2!\,2!} = \binom{14}{3}\binom{11}{3}\binom{8}{2}\binom{6}{2}5! = 605\,404\,800$$

U S C L

   (c) POSSESSIVENESS

   👈 Can you think of a word with more s's?

$$\frac{14!}{6!\,3!} = 20\,180\,160$$

S E

2. Compute $\binom{4}{k}$ for each $k \in \{0, 1, 2, 3, 4\}$.

$0! = 1$

$$\binom{4}{0} = \frac{4!}{4!\,0!} = 1, \quad \binom{4}{1} = \frac{4!}{3!\,1!} = 4, \quad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

$$\binom{4}{3} = \frac{4!}{1!\,3!} = 4, \quad \binom{4}{4} = \frac{4!}{4!\,0!} = 1$$

3. Draw a copy of Pascal's triangle, at least through the row corresponding to $n = 4$. How does this relate to your answers to #2?

k=0, k=1, k=2, k=3, k=4

n=0 → 1

n=1 → 1 1

n=2 → 1 2 1

n=3 → 1 3 3 1

n=4 → 1 4 6 4 1

n=5 → 1 5 10 10 5 1

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\vdots$$
$$\binom{n}{k}$$

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$$

4. Use Pascal's formula (several times) to derive a formula for $\binom{n+3}{r}$ in terms of values of $\binom{n}{k}$ with $k \le r$.

$$\binom{n+3}{r} = \binom{n+2}{r-1} + \binom{n+2}{r}$$
$$= \binom{n+1}{r-2} + \binom{n+1}{r-1} + \binom{n+1}{r-1} + \binom{n+1}{r}$$
$$= \binom{n}{r-3} + \binom{n}{r-2} + \binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r-1} + \binom{n}{r}$$
$$= 1\binom{n}{r-3} + 3\binom{n}{r-2} + 3\binom{n}{r-1} + 1\binom{n}{r}$$

You may assume n and r are integers with $n \ge r \ge 3$.

5. Expand $(x+y)^5$. How are combinations involved here?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y) = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$$
$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$\boxed{\text{BINOMIAL THEOREM: } (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

6. (a) Verify each of the following identities:

$$3^2 = 2^2 + 2 \cdot 2 + 1$$

$$3^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$$

$$3^4 = 2^4 + 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$

$$3^5 = 2^5 + 5 \cdot 2^4 + 10 \cdot 2^3 + 10 \cdot 2^2 + 5 \cdot 2 + 1$$

Do you see the binomial coefficients?

(b) The verifications in part (a) seem to suggest the following identity:

$$3^n = 2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \binom{n}{3}2^{n-3} \cdots + \binom{n}{n-1}2 + 1$$

Express this identity using summation notation, and show how it follows from the Binomial Theorem.

$$3^n = (\underset{a}{2} + \underset{b}{1})^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \cdot 1^k$$

7. Prove that $\sum_{i=0}^{n} (-1)^i \binom{n}{i} 3^{n-i} = 2^n$ for all integers $n \geq 0$.

Binomial theorem with $a = 3$, $b = -1$:

$$2^n = (3-1)^n = \sum_{i=0}^{n} \binom{n}{i} 3^{n-i} (-1)^i$$

8. Use the binomial theorem to evaluate the sum: spicy!

$$\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{2^2}\binom{n}{2} - \frac{1}{2^3}\binom{n}{3} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}$$

Use the Binomial Theorem with $a = 1$ and $b = -\frac{1}{2}$:

$$\left(1 - \tfrac{1}{2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{(1)^{n-k}}_{1} \left(-\tfrac{1}{2}\right)^k$$

← this is almost the sum we want, but it has an extra term

simplify

$$\left(\tfrac{1}{2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(-\tfrac{1}{2}\right)^k$$

separate out the last term

$$\left(\tfrac{1}{2}\right)^n = \left(\sum_{k=0}^{n-1} \binom{n}{k} \left(-\tfrac{1}{2}\right)^k\right) + \underbrace{\binom{n}{n}}_{1} \left(-\tfrac{1}{2}\right)^n$$

↑ This is the sum that we want!

$$\left(\tfrac{1}{2}\right)^n - \left(-\tfrac{1}{2}\right)^n = \sum_{k=0}^{n-1} \binom{n}{k} \left(-\tfrac{1}{2}\right)^k$$

Therefore, the given sum is:

$$\sum_{k=0}^{n-1} \binom{n}{k} \left(-\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}\right)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2^{n-1}} & \text{if } n \text{ is odd.} \end{cases}$$

9. **BONUS:**

(a) Choose several rows of Pascal's triangle. Add up the numbers in each row. What do you notice?

1 — sum is 1

1 1 — sum is 2

1 2 1 — sum is 4

1 3 3 1 — sum is 8

1 4 6 4 1 — sum is 16

1 5 10 10 5 1 — sum is 32

The sum is always a power of 2!

(b) Based on your observations in part (a), what do you think $\sum_{k=0}^{n} \binom{n}{k}$ equals?

It appears that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(c) How does your answer to part (c) relate to the power set of a set of n elements?

Let S be a set of n elements.

Then $\mathcal{P}(S)$ has 2^n elements, which are subsets of S.

There are: $\binom{n}{0} = 1$ subsets with 0 elements,

$\binom{n}{1} = n$ subsets with 1 element,

$\vdots$

and more generally $\binom{n}{k}$ subsets of k elements for any integer $0 \leq k \leq n$.

Thus the total number of subsets is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$,

which must be 2^n.

See also: Example 9.6.7 in the text.