Math 234

Counting: The Pigeonhole Principle Day 16

1. In any set of 27 English words, can you always find two that start with the same letter? Why or why not?

Yes, by the Pigeonhole Principle.

2. If five distinct numbers are chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, must at least one of them be prime? Why?

No, for example: $\{1, 4, 6, 8, 9\}$

3. What is the smallest positive integer N such that any group of N people *must* include at least two with the same birthday? (month and day, not necessarily year)

Including Feb. 29, there are 366 possible birthdays.

$N = 367$

4. Suppose there are 3040 students currently enrolled at St. Olaf College. What is the ~~smallest~~ largest number N of students such that N must share a birthday?

$\frac{3040}{366} = 8.3\ldots$ $N = 9$

5. You and two friends have agreed to score papers for an essay contest. The three of you have decided to use pens of the same color to grade the essays but the group doesn't care what the common color is. You stop by the MSCS department office to find suitable pens, where you are presented with a bag containing a large number of pens in four different colors. If you must pick from the bag without looking at the pens first, what is the largest number of draws you might have to make before successfully finding three pens of the same color?

until

9 draws

n choose k:
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

num. ordered arrangements: $\underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} = 840$

num. unordered arrangements: $\longrightarrow \frac{840}{4 \cdot 3 \cdot 2 \cdot 1} = 35$

6. Given there are seven candidates for membership on a four-person board.

   (a) How many different board rosters are possible?

   "7 choose 4" $\longrightarrow$ binomial coefficient $\rightarrow$ $\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$

   (b) Suppose the candidate receiving the most votes is designated as the board's chair with special powers. The remaining three members of the board do not have any special powers. How many different board configurations are possible as a result of the election?

   7 possible chairs, then choose 3 of 6 to complete the board.

   possible configurations: $7 \cdot \binom{6}{3} = 7 \cdot \frac{6!}{3!(6-3)!} = 7 \cdot \frac{6!}{3! \; 3!} = 140$

7. Consider the set S of integers from one to twenty, inclusive. Pick four distinct numbers from this set.

   (a) What is the smallest possible total for these four numbers?

   10

   (b) What is the largest possible total for these four numbers?

   74

   (c) How many different totals are possible?

   10, 11, 12, ..., 74 — 65 different totals

   (d) How many ways can four distinct numbers be chosen from the set S when order does not matter? (Order doesn't matter because we will be adding these numbers together.)

   $$\binom{20}{4} = \frac{20!}{4! \, 16!} = 4845$$

   (e) Is it true that for some total T there are at least 75 different subsets of four numbers in S that add up to T? Explain why or why not.

   Yes! 4845 pigeons (subsets) in 65 pigeonholes (totals), so some pigeonhole (total) has at least 75 pigeons (subsets).

   $\frac{4845}{65} = 74.5\ldots$

8. In a certain lottery, players select six numbers from 1 to n. For each drawing, balls numbered 1 to n are placed in a hopper, and six balls are drawn at random and without replacement. To win, a player's numbers must match those on the balls, in any order.

   (a) If $n = 15$, how many combinations of winning numbers are possible?

   $$\binom{15}{6} = 5005$$

   (b) If $n = 24$, how many combinations of winning numbers are possible?

   $$\binom{24}{6} = 134596$$

   (c) If $n = 24$, what is the probability that the six balls that are drawn contain only numbers less than 16?

   $$\frac{\binom{15}{6}}{\binom{24}{6}} = \frac{5005}{134596} = 0.037$$

   (d) If $n = 24$, what is the probability that the ball numbered 8 is among the balls drawn?

   $$\frac{1 \cdot \binom{23}{5}}{\binom{24}{6}} = \frac{1}{4}$$

9. In how many ways can 12 distinct books be distributed among four (distinct) children so that

   (a) Each child receives three books?

   $$\underbrace{\binom{12}{3}}_{\text{first child}}\underbrace{\binom{9}{3}}_{\text{second child}}\underbrace{\binom{6}{3}}_{\text{third child}}\underbrace{\binom{3}{3}}_{\text{fourth child}} = 369\,900$$

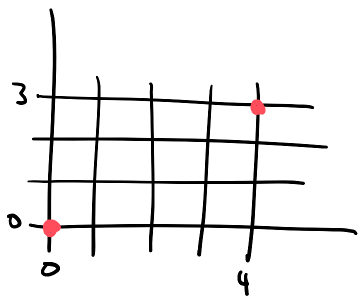
   (b) The two oldest children receive four books each, while the two youngest children receive two books each?

   $$\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = 207\,900$$

10. Consider the 20 "integer lattice points" (a, b) in the xy-plane given by $0 \le a \le 4$ and $0 \le b \le 3$, with a and b integers. Suppose you want to walk along the lattice points from $(0,0)$ to $(4,3)$, and the only legal steps are one unit to the *right* or one unit *up*.

Draw a little picture.

(a) How many legal paths are there from $(0,0)$ to $(4,3)$?

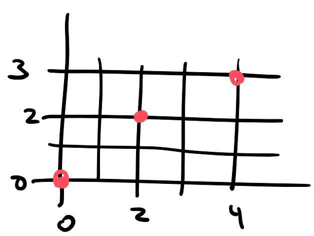


7 steps are required.

Choose any 4 of seven steps to be up, the other 3 must be to the right.

Number of paths: $\binom{7}{4} = 35$

(b) How many legal paths from $(0,0)$ to $(4,3)$ go through the point $(2,2)$?



$$\binom{4}{2}\binom{3}{1} = 18$$

paths from (0,0) to (2,2)

paths from (2,2) to (4,3)

(c) Generalize! How many legal paths are there from $(0,0)$ to (n,k)?

$n+k$ steps, any k of which must be up, and the other n steps to the right.

Thus, the number of paths is $\binom{n+k}{n}$.