

Math 234

Counting: The Addition Rule

Day 15

1. Consider the letters in the word **QUARTILE**.

(a) In how many ways can these eight letters be arranged?

$$\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 8! = 40\,320$$

MULTIPLICATION RULE

PERMUTATION

(b) In how many ways can these eight letters be arranged if the **QU** must remain next to each other (in that order)?

$$7!$$

(c) In how many ways can these eight letters be arranged such that the two-letter sequence **QU** does *not* appear?

$$\begin{array}{ccc} 8! & - & 7! = 35\,280 \\ \text{total} & & \text{with QU} \quad \text{sequences without QU} \end{array}$$

ADDITION RULE
or
DIFFERENCE RULE

(d) How many arrangements of these eight letters contain the substring **ARE**?

$$6! = 720$$

(e) How many arrangements of these eight letters contain both substrings **QU** and **ARE**?

$$5! = 120$$

T, I, L

(f) How many arrangements of these eight letters start with a vowel?

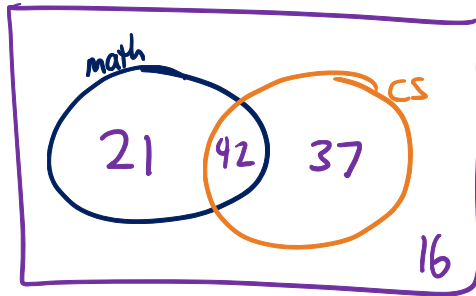
$$\begin{array}{ccc} 4 \cdot 7! & = & 20\,160 \\ \uparrow & \uparrow & \\ \text{vowels} & \text{other letters} & \end{array} \quad = \frac{8!}{2} \quad \begin{array}{l} \text{half of the letters} \\ \text{are vowels} \end{array}$$

(g) If these letters are arranged at random, what is the probability that the arrangement starts with a vowel?

$$\frac{20\,160}{40\,320} = \frac{4 \cdot 7!}{8!} = \frac{4 \cdot \cancel{7!}}{8 \cdot \cancel{7!}} = \frac{1}{2}$$

2. A tech firm receives 116 job applicants. Of these, 63 majored in math, 79 majored in computer science, and 42 majored in both math and computer science. How many of the applicants majored in neither math nor computer science?

16



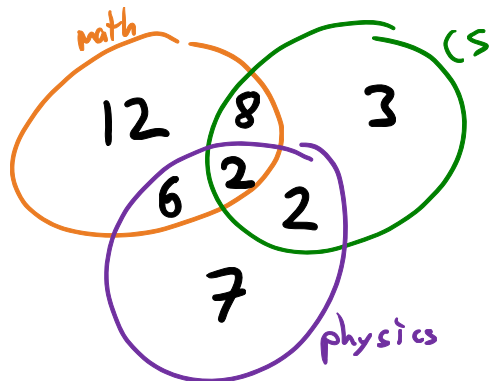
3. Generalize the previous problem. If you know the number of items in sets A , B , and $A \cap B$, then how many items are in $A \cup B$?

$$N(A) + N(B) - N(A \cap B) = N(A \cup B)$$

number of elements

INCLUSION-
EXCLUSION
PRINCIPLE

4. In a certain class, every student is majoring in at least one of the subjects math, computer science, and physics. Suppose that 28 students are majoring in math (and possibly other majors), 15 in computer science (and possibly other majors), 17 in physics (and possibly other majors), 10 in math and CS (and possibly another major), 8 in math and physics (and possibly another major), 4 in computer science and physics (and possibly another major), and 2 in all three majors. How many students are in the class?



Total: 40

5. Generalize the previous problem. If you know the number of items in sets A , B , C , $A \cap B$, $A \cap C$, $B \cap C$, and $A \cap B \cap C$, then how many items are in $A \cup B \cup C$?

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

$$40 = 28 + 15 + 17 - 10 - 8 - 4 + 2$$

6. How many positive integers not exceeding 100 are divisible by either 4 or 6?

A: integers ≤ 100 that are divisible by 4 $N(A) = 25$

B: integers ≤ 100 that are divisible by 6 $N(B) = 16$

$A \cap B$: integers ≤ 100 that are divisible by 4 and 6 $N(A \cap B) = 8$

$$N(A \cup B) = 25 + 16 - 8 = 33$$

7. How many positive integers not exceeding 200 are divisible by either 4, 5, or 6?

$$50 + 40 + 33 - 10 - 16 - 6 + 3 = \boxed{94}$$

↑ multiples of 4 ↑ multiples of 5 ↑ multiples of 6
 ↑ multiples of 4 and 5 ↑ multiples of 4 and 6 ↑ multiples of 5 and 6
 ↑ multiples of 4, 5, and 6

Bonus!

8. Two integers a and b are **relatively prime** if their greatest common divisor is 1. How many positive integers less than 1000 are relatively prime to 1000?

We want to count the number of integers in $\{1, 2, 3, \dots, 999\}$ that are divisible by neither 2 nor 5.

499 of the integers are divisible by 2

199 of the integers are divisible by 5

99 of the integers are divisible by 2 and 5

$$999 - (499 + 199 - 99) = 999 - 599 = 400$$

400 positive integers ≤ 1000 are relatively prime to 1000

Bonus!

9. Let p and q be distinct primes, and let $n = pq$. How many positive integers less than n are relatively prime to n ?

Consider the integers $\{1, 2, 3, \dots, pq-1\}$. ← $pq-1$ integers

$q-1$ of these integers are multiples of p .

$p-1$ of these integers are multiples of q .

None of these integers are multiples of pq .

The number of integers in $\{1, 2, 3, \dots, pq-1\}$ that are relatively prime to pq is:

$$\begin{aligned}(pq-1) - (p-1) - (q-1) &= pq - p - q + 1 \\ &= (p-1)(q-1)\end{aligned}$$