Math 234

Set Theory Day 13

Discuss the following problems with the people at your table.

1. The following is an incomplete proof that for all sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$. Fill in the blanks to complete the proof.

   Proof: Suppose A and B are any sets and $A \subseteq B$.

   Let $x \in$ $A \cup B$.

   By definition of union, $x \in$ A or $x \in$ B.

   In the first case, since $A \subseteq B$, $x \in$ B.

   In the second case it is clear that $x \in B$.

   Thus, in either case, $x \in$ B, which is what we needed to show.

2. Prove that for all sets A and B, $A \cap B \subseteq A$.

   Suppose A and B are any sets and $x \in A \cap B$.

   By definition of intersection, $x \in A$ and $x \in B$.

   In particular, $x \in A$, which is what we wanted to show.

   Therefore, $A \cap B \subseteq A$.

3. Consider sets A and B:

$$A = \{m \in \mathbf{Z} \mid m = 5a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 10b - 15 \text{ for some integer } b\}$$

Prove that $B \subseteq A$.

Suppose $x \in B$. Then $x = 10b - 15$ for some integer b.
So $x = 5(2b-3)$. Let $a = 2b-3$, which is an integer.
Thus, $x = 5a$, which means that $x \in A$.

Then $x \in A$.

4. Prove that for all sets A and B, $(B - A) = B \cap A^c$.

Let A and B be any sets.

(1) Show that $B - A \subseteq B \cap A^c$.

Let $x \in B - A$. So $x \in B$ and $x \notin A$.
Since $x \notin A$, then $x \in A^c$ by definition of complement.
By definition of intersection, if $x \in B$ and $x \in A^c$, then $x \in B \cap A^c$.

(2) Show that $B \cap A^c \subseteq B - A$.

Let $x \in B \cap A^c$. Then $x \in B$ and $x \in A^c$.
Since $x \in A^c$, $x \notin A$.
Then $x \in B$ and $x \notin A$, so $x \in B - A$.

5. Prove the following theorem by induction on n, the number of sets in the union.

Theorem: Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets, and let B be a set. If $A_i \subseteq B$ for all integers $i \geq 1$, then

$$\left(\bigcup_{i=1}^{n} A_i\right) \subseteq B$$

for every positive integer n.

First write the statement $P(n)$ that needs to be proved.

Suppose $A_i \subseteq B$ for all integers $i \geq 1$.

Let $P(n)$ be the statement

$$\left(\bigcup_{i=1}^{n} A_i\right) \subseteq B$$

Basis case: $A_1 \subseteq B$

This is true by assumption.

$P(1)$: $\left(\bigcup_{i=1}^{1} A_i\right) \subseteq B$

$A_1 \subseteq B$

Induction: Suppose $P(k)$ is true for some integer k. That is, $\left(\bigcup_{i=1}^{k} A_i\right) \subseteq B$. } induction hypothesis

Also, $A_{k+1} \subseteq B$ by assumption.

Then, $\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1} \subseteq B$, which is the same as $\bigcup_{i=1}^{k+1} A_i \subseteq B$.

So $P(k+1)$ is true.

6. Consider the statement: For all sets A and B, if $A \subseteq B$, then $A \cap B^c = \emptyset$.

Write the negation of the statement. Then prove the statement by contradiction.

NEGATION: There exist sets A and B with $A \subseteq B$ and $A \cap B^c \neq \emptyset$.

PROOF: Suppose the negation of the statement is true.
Then let sets A and B be such that $A \subseteq B$ and $A \cap B^c \neq \emptyset$.
Since $A \cap B^c$ is nonempty, there exists some $x \in A \cap B^c$.
Now $x \in B^c$ implies that $x \notin B$.
So $x \in A$ and $x \notin B$, which means that A is <u>not</u> a subset of B. This is a contradiction.
Thus, the negation is false and the original statement is true.

7. Prove that for all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Suppose not. Then there exist sets A, B, C with $B \cap C \subseteq A$ and $(C-A) \cap (B-A) \neq \emptyset$.
Since $(C-A) \cap (B-A)$ is not empty, there exists some $x \in (C-A) \cap (B-A)$.
So $x \in C-A$ and $x \in B-A$.
Since $x \in C-A$, $x \in C$ and $x \notin A$.
Since $x \in B-A$, $x \in B$ and $x \notin A$.
This implies $x \in B \cap C$ and $x \notin A$, but then $B \cap C$ cannot be a subset of A.
This is a contradiction. Thus the negation must be false and the original statement is true.