Math 234

Recursive Sequences Day 11

Discuss the following problems with the people at your table.

$a_n = a_{n-1} + 3$

1. Let $a_1 = 5$ and [redacted] for integers $n \geq 2$.

   (a) Write the first five terms of the sequence a_1, a_2, a_3, a_4, a_5.

   $a_2 = a_1 + 3 = 5 + 3 = 8$

   $a_3 = a_2 + 3 = 8 + 3 = 11$

   $a_4 = 11 + 3 = 14$

   $a_5 = 14 + 3 = 17$

   1 2 3 4 5 index

   5, 8, 11, 14, 17

   ☜ What kind of sequence is this?

   ARITHMETIC SEQUENCE

   (b) Can you think of a closed form expression (that is, a non-recursive formula) for a_n?

   $a_n = 3n + 2$

2. Let $b_1 = 4$ and $b_n = \frac{3}{2}b_{n-1}$ for integers $n \geq 2$.

   $b_3 = \frac{3}{2} b_2 = \left(\frac{3}{2}\right) \cdot 4\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2$

   (a) Write the first five terms of the sequence b_1, b_2, b_3, b_4, b_5.

   $b_1 = 4, \quad b_2 = 4\left(\frac{3}{2}\right) = \frac{12}{2}, \quad b_3 = 4\left(\frac{3}{2}\right)^2, \quad b_4 = 4\left(\frac{3}{2}\right)^3, \quad b_5 = 4\left(\frac{3}{2}\right)^4$

   common ratio $\frac{3}{2}$

   ☜ What kind of sequence is this?

   GEOMETRIC SEQUENCE

   (b) Can you think of an explicit (that is, non-recursive) formula for b_n?

   $b_n = 4\left(\frac{3}{2}\right)^{n-1}$

3. Let $c_1 = 3$ and $c_n = c_{n-1} + 2n - 1$ for integers $n \geq 2$.

   (a) Write the first five terms of the sequence c_1, c_2, c_3, c_4, c_5.

   $c_1 = 3, \quad c_2 = 3 + 2(2) - 1 = 6, \quad c_3 = 6 + 2(3) - 1 = 11$

   $c_4 = 11 + 2(4) - 1 = 18, \quad c_5 = 18 + 2(5) - 1 = 27$

   (b) Can you think of a closed form expression for c_n?

   $c_n = n^2 + 2$

multiply by 3

4. Consider the sequence $5, 15, 45, 135, 405, \ldots$. Find both a recurrence relation and an explicit formula for this sequence.

recursive: $a_1 = 5$ and $a_n = 3a_{n-1}$ for $n \geq 2$

closed-form: $a_n = 5(3)^{n-1}$ for $n \geq 1$

or: $a_n = 5(3)^n$ for $n \geq 0$

5. Consider the sequence $1, 11, 101, 1001, 10001, 100001, \ldots$. Find both a recurrence relation and an explicit formula for this sequence.

explicit formula: $a_n = 10^n + 1$ for $n \in \{0, 1, 2, \ldots\}$

one way to write the recurrence:

$$a_1 = 1$$
$$a_n = 10(a_{n-1} - 1) + 1 \quad \text{for } n \geq 1$$

6. Consider the sequence $0, 2, 8, 26, 80, 242, \ldots$. Find both a recurrence relation and an explicit formula for this sequence.

cubes?

explicit formula: $a_n = 3^n - 1$ for $n \in \{0, 1, 2, 3, \ldots\}$

recurrence:

$$a_0 = 0$$
$$a_n = 3a_{n-1} + 2 \quad \text{for } n \geq 1$$

$a_n = a_{n-1} + 2^n$

7. Let $a_0, a_1, a_2, \ldots$ be defined by the formula $a_n = 2^{n+1} - 1$ for all integers $n \geq 0$. Prove that this sequence satisfies the recurrence relation $a_n = a_{n-1} + 2^n$.

What is the formula for a_{n-1}?

write formula for a_{n-1}: $a_{n-1} = 2^{(n-1)+1} - 1$

$a_{n-1} = 2^n - 1$

plug into the right side of recurrence relation: $a_{n-1} + 2^n = (2^n - 1) + 2^n$

$= 2 \cdot 2^n - 1$

$= 2^{n+1} - 1 = a^n$

Thus, $a_{n-1} + 2^n = a_n$.

8. Let $b_0, b_1, b_2, \ldots$ be defined by the formula $b_n = \frac{n}{n+1}$ for all integers $n \geq 1$. Prove that this sequence satisfies the recurrence relation $b_n = b_{n-1} + \frac{1}{n(n+1)}$.

First, by definition $b_{n-1} = \frac{n-1}{(n-1)+1} = \frac{n-1}{n}$

Plug into recurrence: $b_{n-1} + \frac{1}{n(n+1)} = \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1) + 1}{n(n+1)}$

$= \frac{n^2 \cancel{-1} \cancel{+1}}{n(n+1)} = \frac{n}{n+1} = b_n$

Thus, $b_n = b_{n-1} + \frac{1}{n(n+1)}$

9. Consider the sequences defined below by an explicit formula and a recursive formula:

$$a_n = n^3 - 3n^2 + 3n \text{ for } n \in \{0, 1, 2, 3, \ldots\}$$

$$b_0 = 0 \text{ and } b_n = b_{n-1} + 1 \text{ for } n \geq 1$$

Decide whether these two sequences are the same. Justify your conclusion.

These sequences start off the same, but are not equal for $n \geq 3$.

$a_0 = 0,\quad a_1 = 1,\quad a_2 = 2,\quad a_3 = 9,\quad a_4 = 28, \ldots$

$b_0 = 0,\quad b_1 = 1,\quad b_2 = 2,\quad b_3 = 3,\quad b_4 = 4, \ldots$

10. Suppose that $a_0 = 2$ and $a_0, a_1, a_2, \ldots$ is a sequence that satisfies $a_k = a_{k-1} + 3$ for all integers $k \geq 1$. Use mathematical induction to prove that $a_n = 2 + 3n$ for all integers $n \geq 0$.

BASE CASE: Let $n=0$. Then $a_0 = 2 + 3(0) = 2$.

INDUCTION: Let $k \geq 0$ be an integer, and suppose that $a_k = 2 + 3k$.

Then $a_{k+1} = a_k + 3 = (2+3k) + 3$

So $a_{k+1} = 2 + 3(k+1)$

Therefore, $a_n = 2 + 3n$ for all integers $n \geq 0$.

11. A computer algorithm executes twice as many operations when it is run with input of size k as when it is run with input of size $k - 1$. (Assume that $k \geq 1$.) When the algorithm is run with input of size 1, it requires 23 operations to complete its task. How many operations will be required to process an input of size 30?

input size	operations
1	23
2	$23(2) = 46$
3	$23(2)^2 = 92$
$\vdots$	$\vdots$
n	$23(2)^{n-1}$
$\vdots$	$\vdots$
30	$23(2)^{29} = 12{,}348{,}030{,}976$