EXAMPLE: $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$

Property we want to show is true for all integers $n \geq 1$:

$$P(n):\ 1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$$

BASIS STEP: Show that $P(1)$ is true.

$P(1)$: $\quad 1=\dfrac{1(\cancel{2})}{\cancel{2}}$ is clearly true since $1=1$.

MATHEMATICAL INDUCTION:

Show that if $P(k)$ is true, then $P(k+1)$ is also true. for any integer $k \geq 1$

Suppose $P(k)$ is true: $1+2+3+\cdots+k=\dfrac{k(k+1)}{2}$

Then:
$$\begin{aligned}(1+2+3+\cdots+k)+(k+1) &= \frac{k(k+1)}{2}+(k+1) && \leftarrow \text{by the inductive hypothesis}\\ &= \frac{k(k+1)+2(k+1)}{2} && \leftarrow \text{by adding fractions}\\ &= \frac{(k+1)(k+2)}{2} && \leftarrow \text{by factoring}\end{aligned}$$

Therefore, $P(k+1)$ is true:

$$1+2+3+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

Thus, $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ holds for all integers $n \geq 1$.

Math 234

Proof by Induction — Day 10

Discuss the following problems with the people at your table.

1. Let's use mathematical induction to prove that for any integer $n \geq 0$,

$$\sum_{i=0}^{n} 2^i = 2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

(a) Define the basis statement $P(0)$ for this proof.

$P(0)$: $2^0 = 2^{0+1} - 1$

(b) Prove the basis statement $P(0)$. (This will be very short.)

$2^0 = 1 = 2^1 - 1$

(c) Write down the statement $P(k)$, the inductive hypothesis.

$P(k)$: $2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$

(d) Write down the statement $P(k+1)$, which is the statement we want to prove.

$P(k+1)$: $2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$

(e) Finish the proof by showing that if $P(k)$ is true, then $P(k+1)$ follows.

Assume $2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ ← $P(k)$

Add 2^{k+1} to both sides:

$$\underbrace{2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1}}_{\text{left side of } P(k+1)} = 2^{k+1} - 1 + 2^{k+1}$$

$$= (2^{k+1})(2) - 1$$

$$= 2^{k+1+1} - 1$$

$$2^0 + 2^1 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$$

Thus, $P(k+1)$ is true.

2. Use mathematical induction to prove that for any integer $n \geq 1$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}.$$

(a) Define the basis statement $P(1)$ for this proof.

$P(1)$: $\frac{1}{1\cdot 2} = \frac{1}{1+1}$

(b) Prove the basis statement $P(1)$.

Yes, since $\frac{1}{2} = \frac{1}{2}$

(c) Write down the statement $P(k)$, the inductive hypothesis.

$P(k)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

(d) Write down the statement $P(k+1)$, which is the statement we want to prove.

$P(k+1)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$

(e) Finish the proof by showing that if $P(k)$ is true, then $P(k+1)$ follows.

Assume $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Add $\frac{1}{(k+1)(k+2)}$ to both sides:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

3. Use mathematical induction to prove that $\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$ for any integer $n \geq 1$.

BASE CASE: Let $n=1$. Then the statement is

$\sum_{i=1}^{1} i \cdot 2^i = 0 \cdot 2^2 + 2$, which is true since both sides equal 2.

INDUCTION: Let $k \geq 1$ be an integer, and suppose

$$\sum_{i=1}^{k} i \cdot 2^i = (k-1)\, 2^{k+1} + 2.$$

Add $(k+1)\,2^{k+1}$ to both sides:

$$\left(\sum_{i=1}^{k} i \cdot 2^i\right) + (k+1)\,2^{k+1} = (k+1)\,2^{k+1} + (k-1)\,2^{k+1} + 2$$

Combining the sums on the left and factoring on the right, we obtain:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+1} + 2$$

Thus, the given statement holds for $n = k+1$.

Therefore, by mathematical induction, the statement holds for all integers $n \geq 1$.

4. Use mathematical induction to prove that for any integer $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Let $P(n)$ be the statement: $3 \mid 2^{2n} - 1$.

BASE CASE: $P(0)$ is the statement $3 \mid 0$, which is true.

INDUCTION: Assume $P(k)$ is true for some integer $k \geq 0$.

That is, $3 \mid 2^{2k} - 1$, so $2^{2k} - 1 = 3m$ for some integer m.

This implies $2^{2k} = 3m + 1$.

Now consider $2^{2(k+1)} - 1$:

$$2^{2(k+1)} - 1 = 2^{2k} \cdot 2^2 - 1 = (3m+1) \cdot 4 - 1 = 12m + 3$$
$$= 3(4m+1)$$

Let $r = 4m+1$, which is an integer.

Then $2^{2(k+1)} - 1 = 3r$, so $3 \mid 2^{2(k+1)} - 1$, and thus $P(k+1)$ is true.

Therefore, by mathematical induction, $3 \mid 2^{2n} - 1$ for all integers $n \geq 0$.

5. Use mathematical induction to prove that for any integer $n \geq 0$, $n(n^2+5)$ is divisible by 6.

Let $P(n)$ be the statement $6 \mid n(n^2+5)$.

BASE CASE: $P(0)$ is the statement $6 \mid 0$, which is true.

INDUCTION: Suppose $P(k)$ is true for some integer $k \geq 0$.

That is, $6 \mid k(k^2+5)$, which means $k(k^2+5) = 6m$ for some integer m.

Consider $(k+1)\left((k+1)^2+5\right)$ and use algebra to replace $k(k^2+5)$ with $6m$:

$$
\begin{aligned}
(k+1)\left((k+1)^2+5\right) &= (k+1)(k^2+2k+6) = (k+1)\left((k^2+5)+(2k+1)\right) \\
&= \underbrace{k(k^2+5)}_{} + \underbrace{k(2k+1) + (k^2+5) + (2k+1)}_{} \\
&= 6m + 3k^2+3k+6 = 6m + 3k(k+1) + 6 \\
&= 6\left(m + \frac{k(k+1)}{2} + 1\right)
\end{aligned}
$$

Since either k or $k+1$ is even, $\frac{k(k+1)}{2}$ is an integer.

Thus, $(k+1)\left((k+1)^2+5\right)$ is a multiple of 6, and the statement $P(k+1)$ is true.

6. **Bonus:** Find a formula in a, r, m, and n for the sum

Geometric Series $\longrightarrow$ $ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$.

Prove that your formula is correct.

Formula: $\displaystyle\sum_{i=0}^{n} ar^{m+i} = ar^m \frac{r^{n+1}-1}{r-1}$ for $a, r \in \mathbb{R}$, $m, n \in \mathbb{Z}^{\geq 0}$, $r \neq 1$.

PROOF:

BASE CASE: If $n=0$, then the formula is

$$ar^m = ar^m \frac{r^1-1}{r-1},$$ which is true.

INDUCTION: Suppose $\displaystyle\sum_{i=0}^{k} ar^{m+i} = ar^m \frac{r^{k+1}-1}{r-1}$ for some integer k.

Add ar^{m+k+1} to both sides:

$$
\begin{aligned}
\left(\sum_{i=0}^{k} ar^{m+i}\right) + ar^{m+k+1} &= ar^m \frac{r^{k+1}-1}{r-1} + ar^{m+k+1} \\
\sum_{i=0}^{k+1} ar^{m+i} &= ar^m \left(\frac{r^{k+1}-1}{r-1} + r^{k+1}\right) \\
&= ar^m \frac{\cancel{r^{k+1}} - 1 + r^{k+2} \cancel{- r^{k+1}}}{r-1} \\
&= ar^m \frac{r^{k+2}-1}{r-1}
\end{aligned}
$$

which shows the formula holds for $n = k+1$.

Therefore, by mathematical induction, the formula holds for all integers $n \geq 0$.