PROOF TECHNIQUES FOR TODAY:

① Proof by cases:

To prove: "If A_1 or A_2 or A_3 or ... or A_n, then C."

It suffices to show:

$$\left.\begin{array}{l} A_1 \to C \\ A_2 \to C \\ A_3 \to C \\ \vdots \\ \text{and } A_n \to C \end{array}\right\}$$

If all of these are true, then the statement is proved.

EXAMPLE: Prove: The square of an odd integer has the form $8m+1$ for some $m \in \mathbb{Z}$.

PROOF: Let n be an odd integer. Either $n = 4k+1$ or $n = 4k+3$ for some $k \in \mathbb{Z}$. We will consider both cases separately.

CASE 1: $n = 4k+1$

Then $n^2 = (4k+1)^2 = 16k^2 + 8k + 1 = 8(2k^2+k) + 1$, which has the desired form.

CASE 2: $n = 4k+3$

Then $n^2 = (4k+3)^2 = 16k^2 + 24k + 9 = (16k^2 + 24k + 8) + 1$

$= 8(2k^2 + 3k + 1) + 1$, which has the desired form

Thus, the square of any odd integer has the form $8m+1$.

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② Proof by contradiction

- Suppose the statement to be proved is false.
- Show that this leads to a contradiction.

stmt: $P(x) \to Q(x)$

neg: $P(x) \wedge \sim Q(x)$

③ Proof by contraposition

- Express statement to be proved as: $\forall x \in D$, if $P(x)$, then $Q(x)$.
- Rewrite as: $\underbrace{\forall x \in D, \text{ if } \sim Q(x), \text{ then } \sim P(x)}_{\text{prove this by direct proof}}$

Math 234

Cases, Contradiction, and Contraposition Day 8

Discuss the following problems with the people at your table.

1. Warm-up problems about quotients and remainders:

   (a) Compute the values of 39 div 15 and 39 mod 15.

   39 div 15 = 2

   39 mod 15 = 9

   $39 = 2(15) + 9$

   remainder r must be $0 \le r < d$

   (b) Compute the values of −27 div 11 and −27 mod 11.

   −27 div 11 = −3

   −27 mod 11 = 6

   $-27 = -3(11) + 6$

   (c) Today is a Monday. What day of the week will it be:

   Two days from now? Wednesday

   100 days from now? Wednesday

   400 days from now? Tuesday

   400 mod 7 = 1

2. Prove that for all integers n, $n^2 - n + 3$ is odd.

   $n(n-1)+3$

   Break this problem into cases.

PROOF: Let $n \in \mathbb{Z}$. Consider 2 cases.

CASE 1: n is odd. This means $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then $n^2 - n + 3 = (2k+1)^2 - (2k+1) + 3 = (4k^2+4k+1) - 2k - 1 + 3$

$= 4k^2 + 2k + 3 = 2(2k^2+k+1) + 1$, which is odd.

CASE 2: n is even. This means $n = 2k$ for some $k \in \mathbb{Z}$.

Then $n^2 - n + 3 = (2k)^2 - (2k) + 3 = 4k^2 - 2k + 3$

$= 2(2k^2 - k + 1) + 1$, which is odd.

Therefore, $n^2 - n + 3$ is odd for all $n \in \mathbb{Z}$.

3. Prove that for any integer n, the quantity $(n^3 - n)(n+2)$ is divisible by 4.

cases?

PROOF: Let $n \in \mathbb{Z}$. Consider two cases: n even and n odd.

CASE 1: If n is even, then $n = 2m$ for some $m \in \mathbb{Z}$.

Then $(n^3 - n)(n+2) = n(n^2-1)(n+2) = (2m)(4m^2-1)(2m+2)$
$= 4(m)(4m-1)(m+1)$,
which is divisible by 4.

CASE 2: If n is odd, then $n = 2m+1$ for some $m \in \mathbb{Z}$.

Then $(n^3-n)(n+2) = n(n^2-1)(n+2) = (2m+1)((2m+1)^2-1)(2m+1+2)$
$= (2m+1)(4m^2+4m+1-1)(2m+3)$
$= 4(2m+1)(m^2+m)(2m+3)$,
which is divisible by 4.

4. Consider the statement: *The cube root of any irrational number is irrational.*

(a) Write the negation of the statement.

There exists some $x \in \mathbb{R}$, $x \notin \mathbb{Q}$ such that $\sqrt[3]{x} \in \mathbb{Q}$.

(b) Show that assuming the negation is true leads to a contradiction.

Let $y = \sqrt[3]{x}$. Since $\sqrt[3]{x} \in \mathbb{Q}$, $y = \frac{a}{b}$ for integers a and b.

Then $y^3 = \frac{a^3}{b^3}$ is rational.

But $y^3 = x$, which is not rational.

Thus we have a contradiction,
so the cube root of any irrational number
must be irrational.

(c) What does this imply about the original statement?

it's true!

5. Let a, b and c be prime numbers greater than two. Use contradiction to prove that $a + b^2 \neq c^2$.

👉 What is the statement we want to prove? What is its negation?

Let a,b,c be prime numbers greater than 2, and suppose that $a+b^2=c^2$.

Since 2 is the only even prime, a, b, and c must be odd.

Since b is odd, b^2 is also odd.

Since a and b^2 are odd, $a+b^2$ is even.

But c is also odd, which means c^2 is odd.

This is a contradiction, since an even number cannot equal an odd number.

Thus, $a+b^2 \neq c^2$.

6. Consider the statement: *The negative of any irrational number is irrational.*

(a) Write this statement in the form "$\forall x$ in D, if $P(x)$ then $Q(x)$." Then write the contrapositive of this statement.

stmt: $\forall x \in \mathbb{R}$, if $\underbrace{x \notin \mathbb{Q}}_{P(x)}$, then $\underbrace{-x \notin \mathbb{Q}}_{Q(x)}$.

Contrapositive: $\forall x \in \mathbb{R}$, if $\underbrace{-x \in \mathbb{Q}}_{\sim Q(x)}$, then $\underbrace{x \in \mathbb{Q}}_{\sim P(x)}$.

(b) Prove the contrapositive.

Let $x \in \mathbb{R}$ such that $-x \in \mathbb{Q}$.

Then $-x = \frac{a}{b}$ for $a, b \in \mathbb{Z}$.

so $x = \frac{-a}{b} \in \mathbb{Q}$.

(c) What does your proof imply about the original statement?

it's true!

7. Using contraposition, prove the following statement: *For all integers m and n, if $m+n$ is even, then either both m and n are even or both m and n are odd.*

STATEMENT: $\forall\, m,n \in \mathbb{Z}$, if $m+n$ is even, then m and n are either both even or both odd.

CONTRAPOSITIVE: $\forall\, m,n \in \mathbb{Z}$, if m and n have different parity, then $m+n$ is odd.

one is even and the other is odd

PROOF: Let m and n be integers with different parity.

it doesn't matter which is even and which is odd, so we can choose

Without loss of generality, let m be even and n be odd.

So $m = 2k$ and $n = 2r+1$ for some integers k and r.

Then $m+n = 2k + (2r+1) = 2(k+r)+1$, and so $m+n$ is odd.

8. Consider the statement: *The sum of any two irrational numbers is irrational.*

(a) What is wrong with the following "proof" of this statement?

Incorrect proof: Let a and b be any two irrational numbers. The number a cannot be written as the quotient of two integers, nor can b. Therefore, the sum $a+b$ cannot be written as the quotient of two integers, so $a+b$ must be irrational.

This statement is not justified!
In fact, it's wrong!

see "jumping to a conclusion" on page 121 in the text

(b) Is the statement true or false? Provide a proof or a counterexample.

Counterexample: $a = \sqrt{2}$ and $b = 2-\sqrt{2}$ are both irrational,

but $a+b = 2$ is rational.