$a \mid b$ means "a divides b"

Math 234

Direct Proof and Counterexample

$a \mid b \iff b = k \cdot a$ for some $k \in \mathbb{Z}$

Day 7

Discuss the following problems with the people at your table.

1. Using the definition of divisibility prove this statement: For all integers a, b, and c, if $a|b$ and $a|c$ then $a|(b+c)$.

Since $a \mid b$, $b = a \cdot k$ for some integer k.

Also $a \mid c$, therefore $\exists\, m \in \mathbb{Z}$ such that $c = a \cdot m$.

Consider $b+c$:

$$b + c = ak + am = a(k+m)$$

Since k and m are integers, so is $k+m$, and we see that $b+c$ is a multiple of a.

Therefore, $a \mid (b+c)$.

[want: $(b+c) = \text{something} \cdot a$]

2. Consider the following statement: "The negative of any multiple of 3 is a multiple of 3."

   (a) Write the statement formally using a quantifier and a variable.

   $\forall\, x \in \mathbb{R}$, if x is a multiple of 3 then $-x$ is a multiple of 3.

   alternatively: $\forall\, x \in \mathbb{Z}, \ldots$

   (b) Determine whether the statement is true or false, justifying your answer.

   TRUE! PROOF: Suppose x is a multiple of 3. Then $x = 3k$ for some integer k. Multiplying by -1, we have $-x = -3k$. Thus $-x = 3(-k)$, which means that $-x$ is also a multiple of 3.

3. For the statements below, determine whether the statement is true or false. If true, prove the statement directly using definitions. If false, provide a counterexample.

(a) The sum of any three consecutive integers is divisible by 3.

PROOF: Let $n \in \mathbb{Z}$. Consider $s = n + (n+1) + (n+2)$, which is the sum of consecutive integers n, $n+1$, and $n+2$.

Then $s = 3n + 3 = 3(n+1)$, which means that s is a multiple of 3.

Therefore, the sum of any three consecutive integers is a multiple of 3.

(b) A sufficient condition for an integer to be divisible by 8 is that it is divisible by 16.

TRUE! PROOF: Let $n \in \mathbb{Z}$ be divisible by 16.

That means $n = 16k$ for some $k \in \mathbb{Z}$.

So $n = 8(2k)$, which implies that n is divisible by 8.

Thus, any integer divisible by 16 is also divisible by 8.

(c) For all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$.

TRUE! PROOF: Let a, b, c be integers such that $ab | c$.

That is, $c = ab \cdot k$ for some $k \in \mathbb{Z}$.

Then $c = a(bk)$, meaning that $a | c$.

Similarly, $c = b(ak)$, meaning that $b | c$.

Thus, $a | c$ and $b | c$.

(d) For all integers a, b, and c, if $a|bc$ then $a|b$ or $a|c$.

FALSE! Here is a counterexample:

$a = 6, \quad b = 4, \quad c = 3$

Observe that $6 | (4 \cdot 3)$ but 6 does not divide 4 or 3 individually.

4. The standard factored form for an integer n is $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$ where k is a positive integer, $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $e_1, e_2, \ldots, e_k$ are positive integers.

   (a) What is the standard factored form of 2022? How about 2023?

   $$2022 = 2 \cdot 3 \cdot 337$$

   $$2023 = 7 \cdot 17^2$$

   (b) If the standard factored form for an integer a is $a = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$, what is the standard factored form of a^3?

   $$a^3 = \left(p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}\right)^3 = p_1^{3e_1} p_2^{3e_2} p_3^{3e_3} \cdots p_k^{3e_k}$$

   (c) Find the least positive integer j such that $2^4 \cdot 5^5 \cdot 11 \cdot 17^2 \cdot j$ is a perfect cube.

   We want each exponent to be a multiple of 3.

   Choose $j = 2^2 \cdot 5 \cdot 11^2 \cdot 17$.

   Then $2^4 \cdot 5^5 \cdot 11 \cdot 17^2 \cdot j = 2^6 \cdot 5^6 \cdot 11^3 \cdot 17^3 = (2^2 \cdot 5^2 \cdot 11 \cdot 17)^3$.

5. How many zeros are at the end of $65^8 \cdot 56^5$? Explain how you can answer this without actually computing the number.

   $$65^8 \cdot 56^5 = 13^8 \cdot 5^8 \cdot 7^5 \cdot 2^{15}$$

   Since we have eight factors of 5 and fifteen factors of 2, we have eight factors of 10, and the product ends in eight zeros.

6. Prove that if n is a nonnegative integer whose decimal representation ends in 0, then $5|n$.

   Let n be a nonnegative integer whose decimal representation ends in 0.

   That means that $n = 10k$ for some $k \in \mathbb{Z}$.

   Then $n = 5(2k)$, which implies that 5 divides n.