Math 234

Direct Proof and Counterexample — Day 6

Discuss the following problems with the people at your table.

1. Assume that m and n are integers.

   (a) Prove that $14m + 6n + 5$ is odd.

   reasoning from definitions

   Assume m and n are integers.

   Consider $14m + 6n + 5$

   Let $k = 7m + 3n + 2$. Note that k is an integer since integers are closed under $+$ and $\cdot$.

   Then $14m + 6n + 5 = 14m + 6n + 4 + 1 = 2(7m + 3n + 2) + 1$
   $= 2k + 1.$

   So $14m + 6n + 5 = \underbrace{2k + 1, \text{ and thus is odd.}}_{\text{by definition of and odd integer}}$

   (b) Prove that $14m + 6n - 10$ is even.

   Suppose that m and n are arbitrarily chosen integers.

   Consider $14m + 6n - 10$.

   By algebra, $14m + 6n - 10 = 2(7m + 3n - 5)$.

   Let $k = 7m + 3n - 5$.

   Then $14m + 6n - 10 = 2k$.

   So $14m + 6n - 10$ is twice some integer, which implies $14m + 6n - 10$ is even (by definition of even)

2. Show by a counterexample that the following statement is false: "For any two prime numbers m and n, the sum $m + n$ is a composite number."

   $\forall$

   Counterexample: If $\underbrace{m = 2,\ n = 5}_{\text{prime}}$, then $m + n = 7$, which is prime.

   Another counterexample: 2 and 3 are primes whose sum is also prime.

3. In this problem you may use the facts that $(-1)^2 = 1$ and $1^k = 1$ for any integer k. Write a formal proof of each statement below:

(a) If n is an even integer, then $(-1)^n = 1$.

Let n be an even integer. Then $n = 2k$ for some integer k.

Consider $(-1)^n$. By algebra,

$$(-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1$$

Thus, $(-1)^n = 1$ for any even integer n.

(b) If n is an odd integer, then $(-1)^n = -1$.

Let n be an odd integer. So $n = 2k+1$ for some integer k.

Then $(-1)^n = (-1)^{2k+1} = (-1)^{2k}(-1) = 1(-1) = -1$,

where we have used the previous result that $(-1)^{2k} = 1$.

Therefore, $(-1)^n = -1$ for any odd integer n.

4. Prove or disprove the statement: "If k is an odd integer and m is an even integer, then $k^2 + m^2$ is odd."

Let k be odd, so $k = 2a+1$ for some integer a.

Let m be even, so $m = 2b$ for some integer b.

Consider k^2+m^2: by algebra,

$$k^2+m^2 = (2a+1)^2 + (2b)^2 = 4a^2+4a+1+4b^2 = 2(2a^2+2a+2b^2)+1$$

Let $r = 2a^2+2a+2b^2$.

Then $k^2+m^2 = 2r+1$, so k^2+m^2 is odd.

5. Is $0.42424242\ldots$ a rational number? Why or why not?

Let $x = 0.424242\ldots$ then $100x = 42.424242\ldots$

Subtract:

$$\begin{array}{rl} 100x &= 42.4242\ldots \\ -\quad x &= 0.4242\ldots \\ \hline 99x &= 42 \end{array}$$

So $x = \frac{42}{99} = \frac{14}{33}$ and thus x is rational.

6. Is 0.123123123… a rational number? Why or why not?

equals $\frac{123}{999} = \frac{41}{333}$

7. Prove the statement: "If k is a rational number and m is a rational number, then $k^2 + m^2$ is a rational number." You may use the fact that if n and j are integers, so is the quantity n^j.

Since k is rational, $k = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

Similarly, $m = \frac{c}{d}$ for some integers c and d with $d \neq 0$.

Then $k^2 + m^2 = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = \frac{a^2}{b^2} + \frac{c^2}{d^2} = \frac{a^2d^2 + c^2b^2}{b^2d^2}$

Note that $a^2d^2 + c^2b^2$ is an integer, and b^2d^2 is a nonzero integer.

Thus, $k^2 + m^2$ is a rational number (def. of rational).

8. Let r and s be arbitrary rational numbers. Decide whether each of the following statements is true or false and provide a proof of your assertion.

(a) $3r + 2s$ is rational. — TRUE

PROOF: Since r and s are rational, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$.

Then $3r + 2s = 3 \cdot \frac{a}{b} + 2 \cdot \frac{c}{d} = \frac{3a}{b} + \frac{2c}{d} = \frac{3ad + 2bc}{bd}$.

Now $3ad + 2bc$ is an integer and bd is a nonzero integer.

Thus, $3r + 2s$ is rational.

(b) $19r - 4s + \frac{r}{s}$ is rational. — TRUE IF $s \neq 0$

PROOF: Since r and s are rational, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$.

Assuming that $s \neq 0$, we have $c \neq 0$.

Let $N = 19r - 4s + \frac{r}{s}$.

Then $N = 19 \cdot \frac{a}{b} - 4 \cdot \frac{c}{d} + \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{19a}{b} - \frac{4c}{d} + \frac{ad}{bc} = \frac{19acd - 4bc^2 + ad^2}{bcd}$

Observe that $19acd - 4bc^2 + ad^2$ is an integer, and bcd is a nonzero integer. Thus, N is rational.

NOTE: if $s = 0$, then $19r - 4s + \frac{r}{s}$ is undefined.

9. Suppose a, b, c and d are integers. Also suppose x is a real number that satisfies the equation

$$\frac{ax+b}{cx+d} = 1.$$

(a) If the condition that $a \neq c$ is added, decide whether x must be rational and prove the correctness of your assertion.

(b) If we know $a = c$, must x be rational? Prove your answer is correct.

(c) Define the following predicates:

$P(a, b, c, d, x)$ is "x solves the equation $\frac{ax+b}{cx+d} = 1$"
$Q(a, c)$ is "$a = c$"
$R(x)$ is "x is rational"

Use formal logic notation to express the statement "If $a = c$ and x solves the equation, then x must be rational." What is the negation of this statement? (In this problem you can assume a, b, c and d are understood to be integers. You needn't express this explicitly.)