Math 234

Predicates and Quantified Statements — Day 4

Discuss the following problems with the people at your table.

1. Let $P(x)$ be the predicate "$\underbrace{x = x^3}$."

   $x = x^3$ is same as $x^3 - x = 0$

   $x(x^2 - 1) = 0$

   $x(x-1)(x+1) = 0$

   $x = 0, 1,$ or -1

   (a) Write $P(1)$ and $P(-2)$.

   $P(1)$: $1 = 1^3$

   $P(2)$: $-2 = (-2)^3$

   (b) If the domain of x is the positive integers, what is the truth set of $P(x)$?

   $\{1\}$

   (c) If the domain of x is all integers, what is the truth set of $P(x)$?

   $\{-1, 0, 1\}$

   How many solutions does the equation have?

2. Let $P(x)$ be the predicate "x can speak Spanish" and let $Q(x)$ be the predicate "x can code in C++." Express each of the following statements in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain of x is all students at St. Olaf College. $D=$

   $\forall$ for all

   $\exists$ there exists

   (a) There is a St. Olaf student who can speak Spanish and code in C++.

   $\exists\, x \in D : P(x) \wedge Q(x)$

   $\exists\, x$ such that $P(x) \wedge Q(x)$

   (b) There is a St. Olaf student who can speak Spanish but cannot code in C++.

   $\exists\, x : P(x) \wedge \sim Q(x)$

   (c) Every student at St. Olaf can either speak Spanish or code in C++.

   $\forall\, x : P(x) \oplus Q(x)$ — exclusive interpretation

   $\forall\, x,\ P(x) \vee Q(x)$ — inclusive interpretation

   (d) No student at St. Olaf can speak Spanish and code in C++.

   $\forall\, x : \sim(Q(x) \wedge P(x))$

   $\forall\, x,\ \sim Q(x) \vee \sim P(x)$

   $\sim \exists\, x : P(x) \wedge Q(x)$

3. Determine the truth value of each statement if the domain of x is all real numbers.

   (a) $\exists x$ such that $x^2 = 2$.

   True: $x = \sqrt{2}$ satisfies $x^2 = 2$, and $\sqrt{2} \in \mathbb{R}$.
   (also $x = -\sqrt{2}$)

   $\exists$ statement is true if I provide one value to show it is true.

   (b) $\forall x$, $x^2 > 0$.

   False: $x = 0$ is a counterexample.

   $\forall$ statement is false if I provide a single counterexample.

   (c) $\exists x$ such that $x^2 - x < -1$.

   False, since min value of $x^2 - x$ is $-\frac{1}{4}$.

   (d) $\forall x$, $(-x)^2 = x^2$.

   True: squaring a negative makes it positive

   $(x)^{2n} = |x|^{2n}$

   need an argument that applies to all $x \in \mathbb{R}$

4. For each of the following statements, translate from the given informal language to formal language using quantifiers, predicate symbols, and variables.

   (a) No matter what number we start with, there is an integer that is greater than that number.

   $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $y > x$.

   (b) There is an integer such that every number less than this integer is negative.

   $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{R}$, if $y < x$, then $y < 0$.

$Q(x) \Rightarrow P(x)$ means that every x in the truth set of $Q(x)$ is also in the truth set of $P(x)$

5. Let $P(x)$ be the predicate "x is a positive number," and let $Q(x)$ be the predicate "there is a positive number that is less than x."

   (a) Is it true that $Q(x) \Rightarrow P(x)$? Why or why not? Yes

   truth set of $Q(x)$ is $\{x \in \mathbb{R} \mid x > 0\}$ is also truth set of $P(x)$

   (b) Is it true that $\underbrace{Q(x) \Leftrightarrow P(x)}$? Why or why not?

   Yes, the truth sets are identical if $x \in \mathbb{R}$ is the domain.

6. For each predicate $P(x)$, come up with a predicate $Q(x)$ such that $P(x) \Rightarrow Q(x)$. Also specify the domain of x. Is it also true that $P(x) \Leftrightarrow Q(x)$?

   (a) $P(x)$ is "x is a quadrilateral."

   $Q(x)$: sum of the angles of x equals $360°$ $\quad P(x) \Leftrightarrow Q(x)$

   $R(x)$: x is a polygon with 4 angles $\quad P(x) \Rightarrow R(x)$

   $S(x)$: " exactly 4 angles $\quad P(x) \Leftrightarrow S(x)$

   (b) $P(x)$ is "x has two prime factors."

   (c) $P(x)$ is "x satisfies $x^2 = 5$."

7. Consider the following argument.

   All lions are fierce.
   Some lions do not drink coffee.
   ∴ Some fierce creatures do not drink coffee.

   ☞ This argument is from Lewis Carroll, who wrote *Alice in Wonderland* and also several works on symbolic logic.

   Let $P(x)$, $Q(x)$, and $R(x)$ be the predicates "x is a lion," "x is fierce," and "x drinks coffee," respectively. Express each statement of the argument in terms of $P(x)$, $Q(x)$, and $R(x)$. Assume the domain of x is all creatures.

   Does the conclusion of the argument follow from the premises?

8. Consider the following argument.

   No ducks are willing to waltz.
   No officers ever decline to waltz.
   All my poultry are ducks.
   ∴ My poultry are not officers.

   ☞ This argument is also from Lewis Carroll!

   Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the predicates "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz" respectively. Express each statement of the argument in terms of $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

   Does the conclusion of the argument follow from the premises?