POWER SERIES SOLUTIONS

power series: $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$

Most functions that we work with have power series.

For example, $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Example: Find power series solution to $\frac{dy}{dt} = 2ty$.

try: $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$

differentiate: $y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$

plug in to the differential equation:

$$\frac{dy}{dt} = 2ty$$

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = 2t\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\right)$$

match up corresponding terms on both sides:

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = 2a_0 t + 2a_1 t^2 + 2a_2 t^3 + 2a_3 t^4 + \cdots$$

$5a_5 t^4$

constants: $\boxed{a_1 = 0}$

t: $2a_2 = 2a_0$ so $\boxed{a_2 = a_0}$

t^2: $3a_3 = 2a_1$ so $a_3 = \frac{2}{3}a_1 = \frac{2}{3}(0) = 0$ so $\boxed{a_3 = 0}$

t^3: $4a_4 = 2a_2$ so $a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0$ $\boxed{a_4 = \frac{1}{2}a_0}$

t^4: $5a_5 = 2a_3$ so $a_5 = \frac{2}{5}a_3 = 0$ so $\boxed{a_5 = 0}$

If n is odd, then $a_n = 0$.

t^5: $6a_6 = 2a_4$ so $a_6 = \frac{1}{3} a_4 = \frac{1}{3}\left(\frac{1}{2} a_0\right) = \frac{1}{6} a_0$ so $\underline{a_6 = \frac{1}{6} a_0}$

Thus: $\underbrace{y(t) = a_0 + a_0 t^2 + \frac{a_0}{2} t^4 + \frac{a_0}{6} t^6}_{\text{approx. solution}} + \cdots + \frac{a_0}{n!} t^{2n} + \cdots$

SAME ☺

Compare to the exact solution:

$\frac{dy}{dt} = 2ty$ has solution $y(t) = ce^{t^2}$.

The power series for ce^{t^2} is:

$$c\left(1 + (t^2) + \frac{(t^2)^2}{2} + \frac{(t^2)^3}{6} + \cdots\right) = \underbrace{c + ct^2 + \frac{c}{2} t^4 + \frac{c}{6} t^6 + \cdots}$$

Power Series: Worksheet Solutions

1. $\frac{dy}{dt} = 3y \qquad y(t) = \sum_{n=0}^{\infty} a_n x^n$

(a) $a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = 3(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)$

const: $a_1 = 3a_0$

t: $2a_2 = 3a_1 \Rightarrow a_2 = \frac{9}{2}a_0$

t^2: $3a_3 = 3a_2 \Rightarrow a_3 = \frac{9}{2}a_0$

t^3: $4a_4 = 3a_3 \Rightarrow a_4 = \frac{27}{8}a_0$

$$\boxed{y(t) = a_0\left(1 + 3t + \frac{9}{2}t^2 + \frac{9}{2}t^3 + \frac{27}{8}t^4 + \cdots\right)}$$

(b) $y(0) = 2 \Rightarrow a_0 = 2$ so $y(t) = 2 + 6t + 9t^2 + 9t^3 + \frac{27}{4}t^4 + \cdots$

(c) $y(t) = y_0 e^{3t} = y_0\left(1 + 3t + \frac{9}{2}t^2 + \frac{27}{6}t^3 + \frac{81}{24}t^4 + \cdots\right)$

2. $\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 1$

$(2a_2 + 6a_3 t + 12a_4 t^2 + \cdots) + t(a_1 + 2a_2 t + 3a_3 t^2 + \cdots) + (a_0 + a_1 t + a_2 t^2 + \cdots) = 1$

const: $2a_2 + a_0 = 1 \Rightarrow a_2 = \frac{1-a_0}{2}$

t: $6a_3 + a_1 + a_1 = 0 \Rightarrow a_3 = \frac{-1}{3}a_1$

t^2: $12a_4 + 2a_2 + a_2 = 0 \Rightarrow a_4 = \frac{-1}{4}a_2 = \frac{a_0 - 1}{8}$

t^3: $20a_5 + 3a_3 + a_3 = 0 \Rightarrow a_5 = \frac{-1}{5}a_3 = \frac{1}{15}a_1$

$$\boxed{y(t) = a_0 + a_1 t + \frac{1-a_0}{2}t^2 - \frac{1}{3}a_1 t^3 + \frac{a_0 - 1}{8}t^4 + \frac{1}{15}a_1 t^5 + \cdots}$$

3. $\frac{dy}{dt} = y + \sin(2t)$

$(a_1 + 2a_2 t + 3a_3 t^2 + \cdots) = (a_0 + a_1 t + a_2 t^2 + \cdots) + \left(2t - \frac{8t^3}{6} + \frac{32t^5}{5!} - \cdots\right)$

const: $a_1 = a_0$ let $a_1 = a_0 = c$

t: $2a_2 = a_1 + 2 \Rightarrow a_2 = \frac{c-2}{2}$

t^2: $3a_3 = a_2 \Rightarrow a_3 = \frac{c-2}{6}$

t^3: $4a_4 = a_3 - \frac{4}{3} \Rightarrow a_4 = \frac{3a_3 - 4}{12} = \frac{\frac{c-2}{2} - 4}{12} = \frac{c-10}{24}$

t^4: $5a_5 = a_4 \Rightarrow a_5 = \frac{c-10}{120}$

$$\boxed{y(t) = c + ct + \frac{c-2}{2}t^2 + \frac{c-2}{6}t^3 + \frac{c-10}{24}t^4 + \frac{c-10}{120}t^5 + \cdots}$$

4. $\dfrac{dy}{dt} = y^2$

$$\left(a_1 + 2a_2 t + 3a_3 t^2 + \cdots\right) = \left(a_0 + a_1 t + a_2 t^2 + \cdots\right)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right)$$

$$= a_0^2 + 2a_0 a_1 t + \left(a_1^2 + 2a_0 a_2\right)t^2 + \left(2a_0 a_3 + 2a_1 a_2\right)t^3 + \cdots$$

const: $a_1 = a_0^2$ let $a_0 = c$, then $a_1 = c^2$

t: $2a_2 = 2a_0 a_1$ $a_2 = a_0 a_1 = c^3$

t^2: $3a_3 = a_1^2 + 2a_0 a_1$ $a_3 = \frac{1}{3}\left(a_1^2 + 2a_0 a_1\right) = c^4$

t^3: $4a_4 = 2a_0 a_3 + 2a_1 a_2$ $a_4 = \frac{1}{4}\left(2a_0 a_3 + 2a_1 a_2\right) = c^5$

$$\boxed{y(t) = c + c^2 t + c^3 t^2 + c^4 t^3 + c^5 t^4 + \cdots}$$

← Geometric Series for $y(t) = \dfrac{c}{1-ct}$