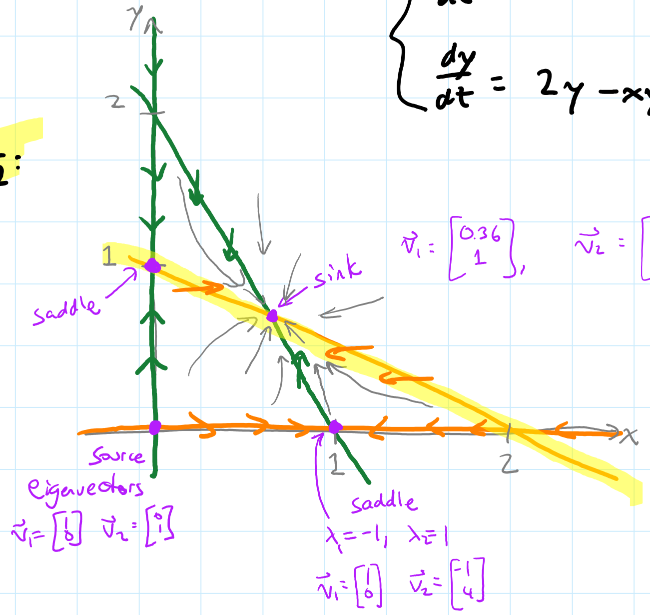
Last time:

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - axy \\[2mm] \dfrac{dy}{dt} = 2y - xy - 2y^2 \end{cases}$$

$a = \frac{1}{2}$:



If $a = 2$:

$$\frac{dx}{dt} = x - x^2 - 2xy = x(1 - x - 2y)$$

$$\frac{dy}{dt} = 2y - xy - 2y^2 = y(2 - x - 2y)$$

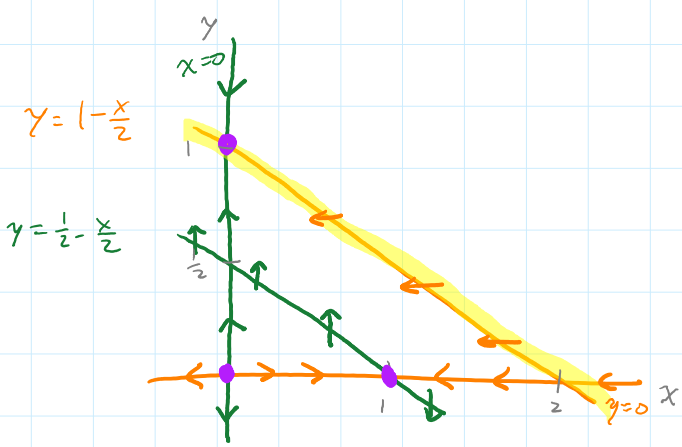
Nullclines:

$x = 0$ or $1 - x - 2y = 0$

$y = \frac{1}{2} - \frac{x}{2}$

$y = 0$ or $2 - x - 2y = 0$

$y = 1 - \frac{x}{2}$



If $x = 0$, $\frac{dx}{dt} = 0$,

and $\frac{dy}{dt} = 2y - 2y^2 = 2y(1 - y)$

If $y = \frac{1}{2} - \frac{x}{2}$, $\frac{dx}{dt} = 0$

and $\frac{dy}{dt} = y(2 - x - 2y)$

$= \left(\frac{1}{2} - \frac{x}{2}\right)\left(2 - x - 2\left(\frac{1}{2} - \frac{x}{2}\right)\right)$

$= \frac{1}{2}(1 - x)(1)$

If $y = 0$, $\frac{dy}{dt} = 0$,

So $\frac{dx}{dt} = x - x^2 = x(1 - x)$

Only one of the nullclines depends on a:

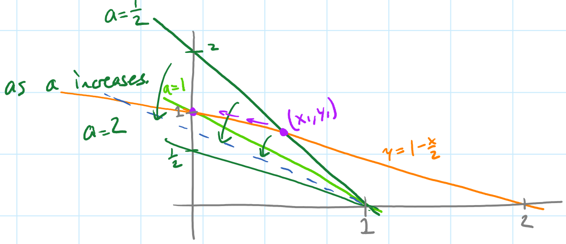
the x-nullcline $1 - x - ay = 0$

$$\underbrace{y = \frac{-x}{a} + \frac{1}{a}}$$

line, slope $\frac{-x}{a}$, y-intercept $\frac{1}{a}$

x-intercept 1

As a increases from $\frac{1}{2}$ to 2, the line gets less steep.



If $0<a<1$, then we have equilibrium points $(0,1)$ and (x_1, y_1) with $x_1>0$, $y_1>0$.

If $a=1$, these two equilibrium points merge, and there is no equilibrium point in the first quadrant.

Thus, a bifurcation occurs at $a=1$.

HAMILTONIAN SYSTEMS (§5.3)

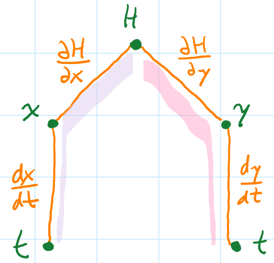
Example:
$$\begin{cases} \frac{dx}{dt} = 2y - 2xy \\ \frac{dy}{dt} = 2x + y^2 \end{cases}$$

Suppose that $(x(t), y(t))$ solves the system.

Let $H(x,y) = y^2 - x^2 - xy^2$.

Claim: $H(x(t), y(t))$ is constant for all t.

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial H}{\partial y} \cdot \frac{dy}{dt}$$
$$= (-2x - y^2)\cdot(2y - 2xy) + (2y - 2xy)(2x + y^2)$$
$$\frac{dH}{dt} = 0$$



Thus, H is constant for all t.

The function $H(x,y)$ is called a conserved quantity of the system.

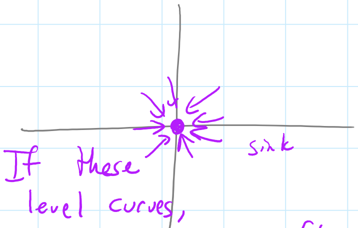
Also, $\frac{dx}{dt} = \frac{\partial H}{\partial y}$ and $\frac{dy}{dt} = -\frac{\partial H}{\partial x}$, so the function is a Hamiltonian function and the system is a Hamiltonian system.

A Hamiltonian function is always a conserved quantity.

Solution curves of the system follow level curves of $H(x,y)$.

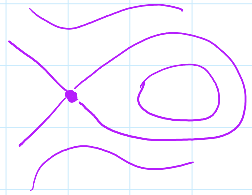
PROPERTY: Hamiltonian systems never have sinks or sources of any type.

why? Geometry of level curves. ⟶



If these are level curves, then the surface is flat.

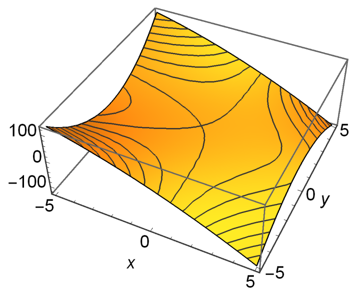
However, Hamiltonian systems can have saddle connection: level curves from a saddle can loop around and connect.



Contour lines on a 3 D surface :

```
In[•]:= Plot3D[y^2 - x^2 - x * y^2, {x, -5, 5},
         {y, -5, 5}, AxesLabel → Automatic, MeshFunctions → {#3 &}]
```

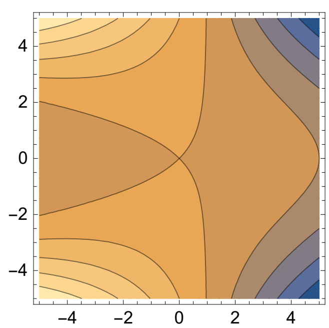
Out[•]=



Contour lines in 2D:

```
In[•]:= c = ContourPlot[y^2 - x^2 - x * y^2, {x, -5, 5}, {y, -5, 5}]
```

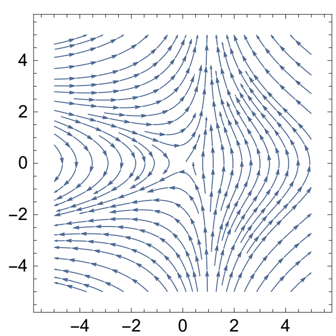
Out[•]=



Phase portrait

```
In[•]:= s = StreamPlot[{2 y - 2 x * y, 2 x + y^2}, {x, -5, 5}, {y, -5, 5}]
```

Out[•]=



Phase portrait and contour plot together:

In[ ]:=
```
Show[c, s]
```

Out[ ]=