OSCILLATOR WITH A PARAMETER

$$y'' + 10y = \cos(\omega t), \qquad y(0) = y'(0) = 0$$

- Last time: $y_h(t) = k_1 \cos(\sqrt{10}t) + k_2 \sin(\sqrt{10}t)$
- If $\omega \neq \sqrt{10}$: $y_p(t) = \frac{1}{10-\omega^2}\cos(\omega t)$

so $y(t) = k_1\cos(\sqrt{10}t) + k_2 \sin(\sqrt{10}t) + \frac{1}{10-\omega^2}\cos(\omega t)$

initial condition: $y(0) = k_1 + 0 + \frac{1}{10-\omega^2} = 0$ so $k_1 = \frac{-1}{10-\omega^2}$

$y'(t) = -\sqrt{10}\,k_1 \sin(\sqrt{10}t) + \sqrt{10}\,k_2\cos(\sqrt{10}t) - \frac{\omega}{10-\omega^2}\sin(\omega t)$

$y'(0) = \quad 0 \quad\quad \sqrt{10}\,k_2 \quad\quad 0$

Since $y'(0) = 0$, $\sqrt{10}\,k_2 = 0$, so $k_2 = 0$

Particular solution: $\boxed{y(t) = \frac{-1}{10-\omega^2}\cos(\sqrt{10}t) + \frac{1}{10-\omega^2}\cos(\omega t)}$

- If $\omega = \sqrt{10}$: $y'' + 10y = \cos(\sqrt{10}t)$

particular solution: try $y_p(t) = At\cos(\sqrt{10}t) + Bt\sin(\sqrt{10}t)$

plug in: $y'' + 10y = \cos(\sqrt{10}t)$

$$2\sqrt{10}\left(B\cos(\sqrt{10}t) - A\sin(\sqrt{10}t)\right) = \cos(\sqrt{10}t)$$

$A = 0$ and $2\sqrt{10}\,B = 1$

$B = \frac{1}{2\sqrt{10}}$

Solution: $y(t) = k_1\cos(\sqrt{10}t) + k_2\sin(\sqrt{10}t) + \frac{1}{2\sqrt{10}}t\sin(\sqrt{10}t)$

Initial conditions: $y(0) = y'(0) = 0 \Rightarrow k_1 = k_2 = 0$

particular solution: $\boxed{y(t) = \frac{t}{2\sqrt{10}}\sin(\sqrt{10}t)}$

FREQUENCIES:

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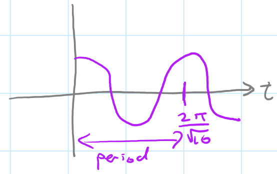


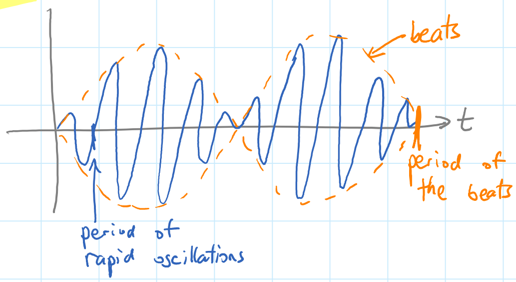
- Natural frequency: frequency of $\underline{\cos(\sqrt{10}\,t)}$
  unforced solution
  is $\frac{1}{\text{period}} = \frac{1}{\frac{2\pi}{\sqrt{10}}} = \frac{\sqrt{10}}{2\pi}$

- Forcing frequency: frequency of $\cos(\omega t)$ is $\frac{\omega}{2\pi}$

- If ω is close to $\sqrt{10}$, then beats emerge

  Frequency of the beats: $\dfrac{\left|\frac{\sqrt{10}}{2\pi} - \frac{\omega}{2\pi}\right|}{2} = \dfrac{|\sqrt{10}-\omega|}{4\pi}$

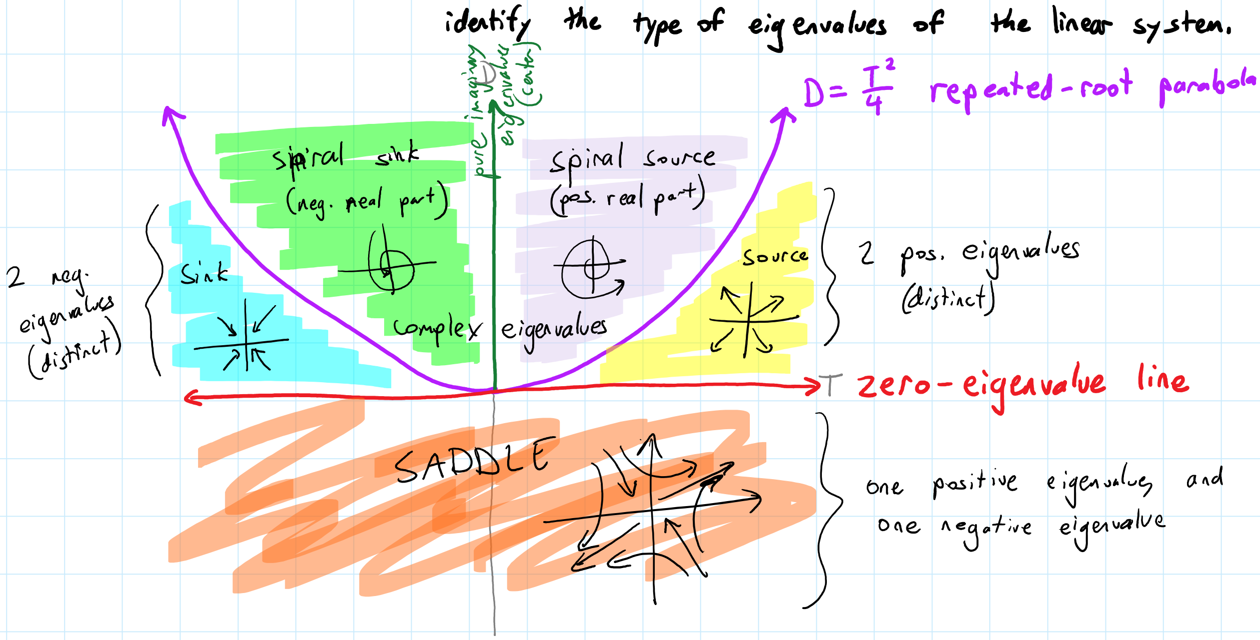
  Frequency of the rapid oscillations: $\dfrac{\frac{\sqrt{10}}{2\pi} + \frac{\omega}{2\pi}}{2} = \dfrac{\sqrt{10}+\omega}{4\pi}$



REVIEW:

Sketch the trace-determinant plane.

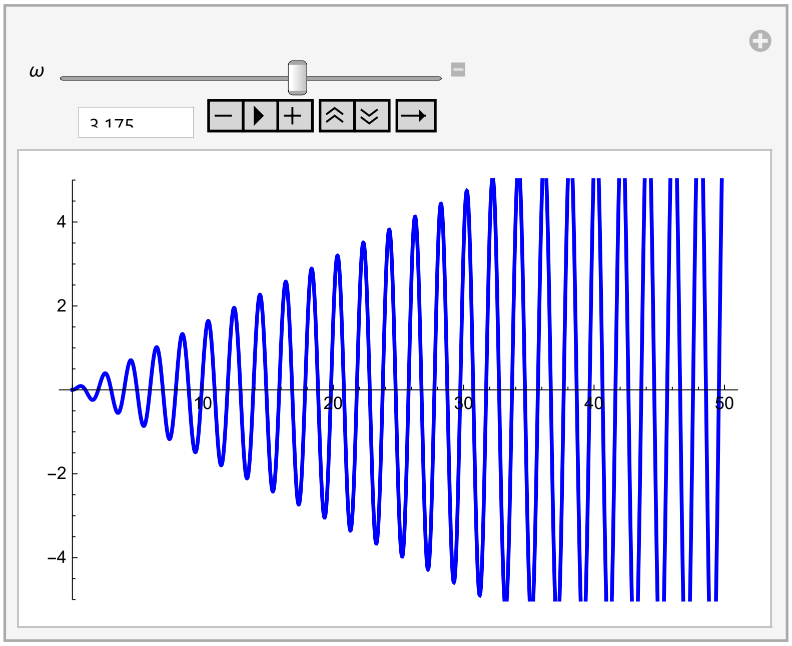
For each region, sketch a typical phase portrait and identify the type of eigenvalues of the linear system.



In[1]:=
```
y[t_, ω_] := -Cos[Sqrt[10] t] / (10 - ω^2) + Cos[ω t] / (10 - ω^2)
```

In[10]:=
```
Manipulate[
 Plot[y[t, ω], {t, 0, 50}, PlotStyle → {Blue, Thick}, PlotRange → {-5, 5}], {ω, 0, 5}]
```

Out[10]=



In[3]:=
```
Sqrt[10.]
```

Out[3]= 3.16228

Find particular solution for case $\omega = $ sqrt(10):

In[4]:=
```
yp[t_] := A t Cos[Sqrt[10] t] + B t Sin[Sqrt[10] t]
```

In[5]:=
```
yp''[t]
```

Out[5]= $2\sqrt{10}\, B \cos\left[\sqrt{10}\, t\right] - 10\, A\, t \cos\left[\sqrt{10}\, t\right] - 2\sqrt{10}\, A \sin\left[\sqrt{10}\, t\right] - 10\, B\, t \sin\left[\sqrt{10}\, t\right]$

In[6]:=
```
Simplify[yp''[t] + 10 yp[t]]
```

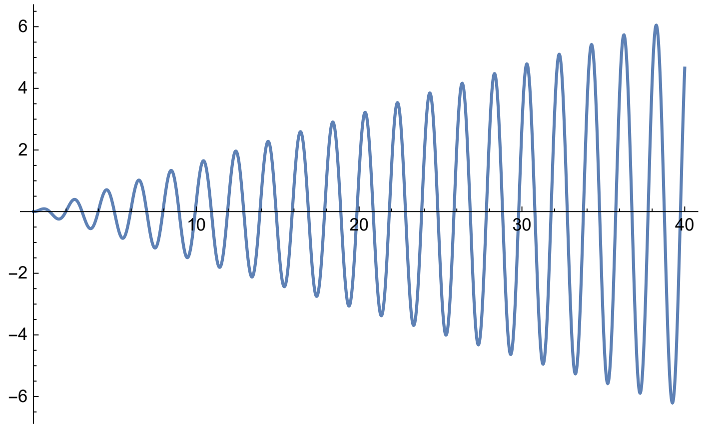
Out[6]= $2\sqrt{10}\left(B \cos\left[\sqrt{10}\, t\right] - A \sin\left[\sqrt{10}\, t\right]\right)$

In[7]:=
```
Solve[yp''[t] + 10 yp[t] == Cos[Sqrt[10] t], {A, B}]
```

Out[7]= $\left\{\left\{B \to \frac{1}{2\sqrt{10}} + A \tan\left[\sqrt{10}\, t\right]\right\}\right\}$

In[9]:=
```
Plot[t Sin[Sqrt[10] t] / (2 Sqrt[10]), {t, 0, 40}]
```

Out[9]=



Plot beats and rapid oscillations

In[11]:=
```
b[t_, ω_] := 2 Sin[(ω - Sqrt[10]) t / 2] / (10 - ω^2)
```

In[12]:=
```
Manipulate[
 Plot[{y[t, ω], b[t, ω], -b[t, ω]}, {t, 0, 50},
  PlotStyle → {{Blue, Thick}, {Orange, Dashed}, {Orange, Dashed}}, PlotRange → {-1, 1}],
 {ω, 0, 5}]
```

Out[12]=