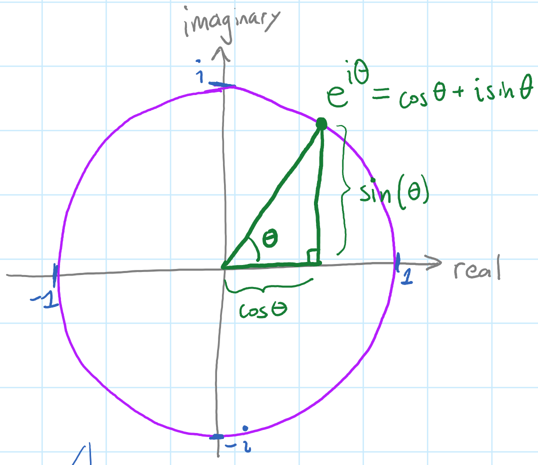
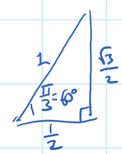
EULER'S FORMULA:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$e^{i\theta}$ maps a real number θ to a point on the unit circle in the complex plane



$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$$



FROM LAST TIME:

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \quad \text{with } A = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}$$

We found that A has eigenvalues $\underbrace{\lambda_1 = -1+i, \quad \lambda_2 = -1-i}_{\text{a conjugate pair}}$

Now, find eigenvectors:

$\lambda_1 = -1+i$, we want $\vec{v}$ such that $(A - \lambda_1 I)\vec{v} = \vec{0}$

$$(A - (-1+i)I)\vec{v} = \begin{bmatrix} 1-(-1+i) & 5 \\ -1 & -3-(-1+i) \end{bmatrix}\vec{v} = \vec{0}$$

$$\begin{bmatrix} 2-i & 5 \\ -1 & -2-i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \qquad \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[-1 \quad -2-i]\cdot(-2+i) = [2-i \quad (-2-i)(-2+i)] = [2-i \quad \underbrace{4-\cancel{2i}+\cancel{2i}-i^2}_{5}]$$

We want: $(2-i)x + 5y = 0$

Let $x = 2+i$, then:

$$(2-i)(2+i) + 5y = 0$$
$$4 + \cancel{2i} - \cancel{2i} - i^2 + 5y = 0$$
$$4 + 1 + 5y = 0$$
$$5 + 5y = 0$$
$$y = -1$$

eigenvector

$$\vec{v}_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = -1+i, \quad \lambda_2 = -1-i$

Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2-i \\ -1 \end{bmatrix}$ ← also come in conjugate pairs

Solutions to $\frac{d\vec{Y}}{dt} = \mathbf{A}\vec{Y}$: $\quad \vec{Y}_1 = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(-1+i)t}, \quad \vec{Y}_2 = \begin{bmatrix} 2-i \\ -1 \end{bmatrix} e^{(-1-i)t}$

We usually want real-valued solutions.

Apply Euler's formula to $\vec{Y}_1(t)$:

$$\vec{Y}_1(t) = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{-t} e^{it}$$

Euler's formula

$$\vec{Y}_1(t) = \begin{bmatrix} 2+i \\ -1 \end{bmatrix} e^{-t} (\cos t + i \sin t)$$

Multiply everything, then separate the real and imaginary parts:

$$\vec{Y}_1(t) = \begin{bmatrix} (2+i)(\cos t + i \sin t) \\ -1\,(\cos t + i \sin t) \end{bmatrix} e^{-t}$$

$$\vec{Y}_1(t) = \begin{bmatrix} 2\cos t + i\,2\sin t + i\cos t - \sin t \\ -\cos t - i \sin t \end{bmatrix} e^{-t}$$

$$\vec{Y}_1(t) = \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^{-t} + i \begin{bmatrix} 2\sin t + \cos t \\ -\sin t \end{bmatrix} e^{-t}$$

$$\vec{Y}_1(t) = \underbrace{\vec{Y}_{re}(t)}_{\text{"real part"}} + i \cdot \underbrace{\vec{Y}_{im}(t)}_{\text{"imaginary part"}}$$

Substitute $\vec{Y}_1(t) = \vec{Y}_{re}(t) + i\,\vec{Y}_{im}(t)$ into $\frac{d\vec{Y}}{dt} = \mathbf{A}\vec{Y}$:

diff.

multiply by A

$$\frac{d\vec{Y}_{re}}{dt} + i\,\frac{d\vec{Y}_{im}}{dt} = \mathbf{A}(\vec{Y}_{re} + i\,\vec{Y}_{im}) = \mathbf{A}\vec{Y}_{re} + i\,\mathbf{A}\vec{Y}_{im}$$

must be the same

must be the same

Thus: $\frac{d\vec{Y}_{re}}{dt} = \mathbf{A}\vec{Y}_{re}$ and $\frac{d\vec{Y}_{im}}{dt} = \mathbf{A}\vec{Y}_{im}$

So $\vec{Y}_{re}$ and $\vec{Y}_{im}$ are the real-valued solutions to the linear system!

For $\frac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} \vec{Y}$, we have solutions:

$$\vec{Y}_{re}(t) = \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} e^{-t} \quad \text{and} \quad \vec{Y}_{im} = \begin{bmatrix} 2\sin t + \cos t \\ -\sin t \end{bmatrix} e^{-t}$$

General solution: $\vec{Y}(t) = k_1 \vec{Y}_{re}(t) + k_2 \vec{Y}_{im}(t)$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Solve: $\frac{d\vec{Y}}{dt} = A\vec{Y}$

Linear Systems with Complex Eigenvalues

Math 230

Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

1. Find the eigenvalues of $\mathbf{A}$. Note that they are complex conjugates.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

so $\lambda = \pm i$

2. Let λ be the eigenvalue with positive imaginary part. Find an eigenvector $\mathbf{V}$ associated with λ.

$\lambda = i$ $\qquad (A - \lambda I)\vec{V} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ implies $-ix + y = 0$.

If $x = i$, then $-i(i) + y = 0$, or $1 + y = 0$, so $y = -1$.

Thus, $\vec{V} = \begin{bmatrix} i \\ -1 \end{bmatrix}$ or any multiple of this vector.

3. Write down the complex-valued "straight-line" solution $\mathbf{Y}_1 = \mathbf{V}e^{\lambda t}$ to the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

$$\vec{Y}_1(t) = \begin{bmatrix} i \\ -1 \end{bmatrix} e^{it}$$

4. Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to expand this solution. Then collect the real and imaginary parts so that $\mathbf{Y}_1 = \mathbf{Y}_{\text{re}} + i\mathbf{Y}_{\text{im}}$, where $\mathbf{Y}_{\text{re}}$ and $\mathbf{Y}_{\text{im}}$ are real-valued functions.

$$\vec{Y}_1(t) = \begin{bmatrix} i \\ -1 \end{bmatrix} e^{it} = \begin{bmatrix} i \\ -1 \end{bmatrix} (\cos t + i \sin t)$$

$$= \begin{bmatrix} i\cos t - \sin t \\ -\cos t - i \sin t \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}}_{\vec{Y}_{re}} + i \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\vec{Y}_{im}}$$

5. Show that $\mathbf{Y}_{\text{re}}$ and $\mathbf{Y}_{\text{im}}$ each satisfy the system $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$.

$$\vec{Y}_{re} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}, \quad \frac{d\vec{Y}_{re}}{dt} = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}, \quad A\vec{Y}_{re} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

Thus, $\frac{d\vec{Y}_{re}}{dt} = A\vec{Y}_{re}$.

$$\vec{Y}_{im} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \frac{d\vec{Y}_{im}}{dt} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}, \quad A\vec{Y}_{im} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

Thus, $\frac{d\vec{Y}_{im}}{dt} = A\vec{Y}_{im}$.

6. Write down the general solution to $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$.

$$\vec{Y}(t) = k_1 \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + k_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

or more simply: $\vec{Y}(t) = c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$.

7. Find a solution with the initial value $\mathbf{Y}(0) = (2, 4)$.

$\vec{Y}(0) = k_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ implies $k_1 = -4, \; k_2 = -2$

Thus, the solution to the initial value problem is

$\vec{Y}(t) = -4 \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} - 2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$ or more simply $\vec{Y}(t) = \begin{bmatrix} 4\sin t + 2\cos t \\ 4\cos t - 2\sin t \end{bmatrix}$

8. What is the long-term behavior of the solution you found?

The solution oscillates.

In the phase plane, the solution is a circle of radius $\sqrt{4^2+2^2} = \sqrt{20}$.