EXAMPLE: What are the eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$?

$$A\vec{v} = \lambda \vec{v} \longrightarrow \underbrace{(A - \lambda I)}_{\text{must have determinant zero}} \vec{v} = \vec{0}$$

λ: eigenvalue; $\vec{v}$: eigenvector

Want a nonzero vector $\vec{v}$

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda + 1 - 4$$

$$0 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \quad \text{so } \lambda_1 = 3 \text{ or } \lambda_2 = -1$$

$\lambda_1 = 3$: $(A - 3I)\vec{v}_1 = \vec{0}$ so $\boxed{\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}} \vec{v}_1 = \vec{0}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+2 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$: $(A + I)\vec{v}_2 = \vec{0}$ so $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \vec{v}_2 = \vec{0}$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

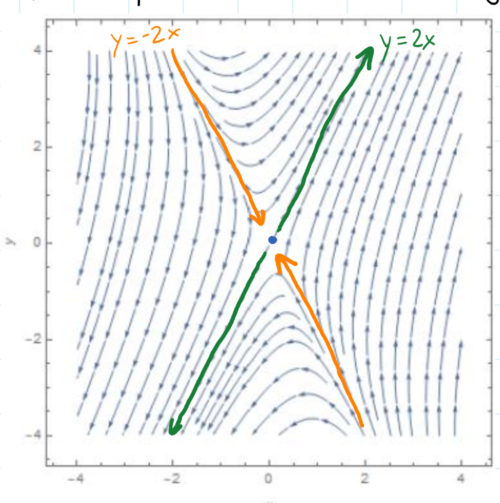
$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+2 \\ -4+4 \end{bmatrix} = \vec{0}$ ✓

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$ and eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

LAST TIME:
$$\begin{cases} \dfrac{dx}{dt} = x + y \\[2mm] \dfrac{dy}{dt} = 4x + y \end{cases} \quad \text{or} \quad \frac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{Y}$$

has solutions $\vec{Y}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$, $\quad \vec{Y}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

Phase portrait of $\dfrac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{Y}$



$\vec{Y}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

$$\begin{cases} x(t) = e^{3t} \\ y(t) = 2e^{3t} \end{cases} \rightarrow \quad y(t) = 2x(t), \quad y = 2x$$

$\vec{Y}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

$$\begin{cases} x(t) = e^{-t} \\ y(t) = -2e^{-t} \end{cases} \quad y(t) = -2x(t), \quad y = -2x$$

Why does $\vec{Y}(t) = \vec{V}e^{\lambda t}$ satisfy $\frac{d\vec{Y}}{dt} = A\vec{Y}$ if $\vec{V}$ is an eigenvector of A corresponding to eigenvalue λ?

- Differentiate: $\frac{d\vec{Y}}{dt} = \vec{V}\lambda e^{\lambda t}$ same! so $\frac{d\vec{Y}}{dt} = A\vec{Y}$
- Multiply by A: $A\vec{Y}(t) = A\vec{V}e^{\lambda t} = \lambda\vec{V}e^{\lambda t}$

WORKSHEET:

$$\frac{dx}{dt} = 2x - 4y$$

$$\frac{dy}{dt} = -x - y$$

eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -2$

eigenvectors: $\vec{V}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solutions: $\vec{Y}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t}$, $\vec{Y}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

general solution: $\vec{Y}(t) = k_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

$\vec{Y}(t) = k_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t} + k_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{-2t}$ SAME

Initial condition:

$$\vec{Y}(0) = \begin{bmatrix} 6 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3(0)} + k_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{-2(0)}$$

or

$$\begin{aligned} 6 &= -4k_1 + 2k_2 \\ -(1 &= 1k_1 + 2k_2) \\ \hline 5 &= -5k_1 \\ -1 &= k_1 \end{aligned}$$

then $1 = \underset{k_1}{(-1)} + 2k_2$

$2 = 2k_2$ or $1 = k_2$

Particular solution: $\vec{Y}(t) = -1\begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t} + 1\begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{-2t}$

$$\vec{Y}(t) = \begin{bmatrix} 4e^{3t} + 2e^{-2t} \\ -e^{3t} + 2e^{-2t} \end{bmatrix}$$

Linear Systems with Real Eigenvalues

Math 230

Consider the linear system:

$$\frac{dx}{dt} = 2x - 4y$$
$$\frac{dy}{dt} = -x - y$$

1. Compute the eigenvalues and eigenvectors of the coefficient matrix for this system.

$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 3$,

with eigenvectors $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{V}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

2. Find two straight-line solutions and the general solution for this system.

$\vec{Y}_1(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

$\vec{Y}_2(t) = k_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$

General solution:

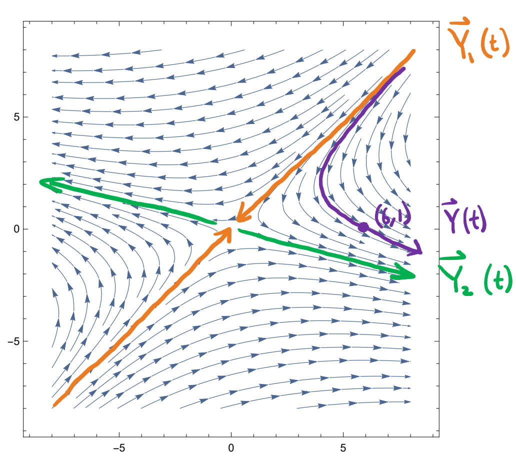
$$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + k_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{3t}$$

3. Compute the solution to the system with initial value $x(0) = 6$, $y(0) = 1$.

$k_1 = 2, \quad k_2 = -1$

$$\vec{Y}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{3t}$$

4. On the plot below, draw the straight-line solutions and the solution to the initial-value problem.



Plot produced in Mathematica by: `StreamPlot[{2x - 4y, -x - y}, {x, -8, 8}, {y, -8, 8}]`

5. Why do the solution trajectories seem to asymptotically approach one of the straight-line solutions, and not the other one?

As $t \to \infty$, the e^{-2t} term of the general solution goes to zero, and the e^{3t} term dominates. Thus, solutions look like the e^{3t} term as t increases.

For each of the matrices below,

(a) Find the eigenvalues and corresponding eigenvectors.

(b) Find the straight-line solutions and the general solution to the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

(c) Match the phase plane with one of the diagrams at the bottom of this page.

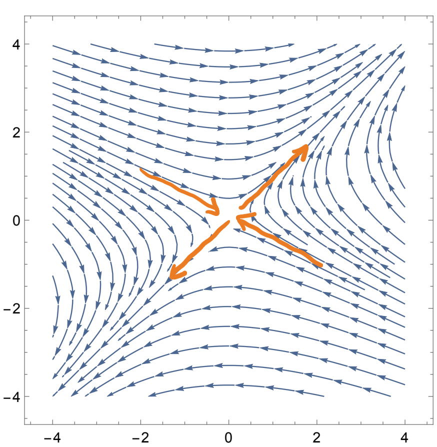
(d) Describe the long-term behavior of solutions to the system.

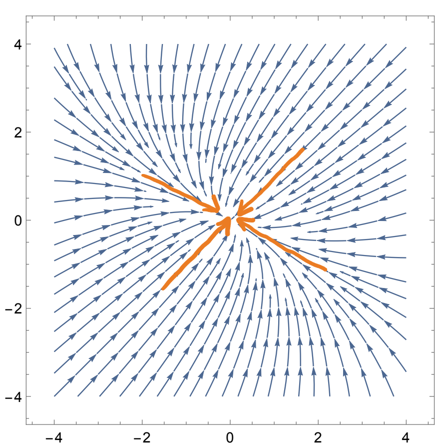
1. $\mathbf{A} = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$ $\lambda_1 = 3,\ \lambda_2 = 6,\quad \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} -3 & 6 \\ 3 & 0 \end{bmatrix}$ $\lambda_1 = -6,\ \lambda_2 = 3,\quad \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} -4 & -2 \\ -1 & -5 \end{bmatrix}$ $\lambda_1 = -6,\ \lambda_2 = -3,\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2





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