COMPLETELY DECOUPLED SYSTEM

1. $\begin{cases} \frac{dx}{dt} = 3x \longrightarrow & x(t) = k_1 e^{3t} \\ \frac{dy}{dt} = y + 2 \longrightarrow & y(t) = k_2 e^{t} - 2 \end{cases}$

so $\frac{dy}{dt} = k_2 e^{t}$

PARTIALLY DECOUPLED SYSTEM

2. $\begin{cases} \frac{dx}{dt} = 3x - 2y \\ \frac{dy}{dt} = 4y \longrightarrow \boxed{y(t) = k_2 e^{4t}} \end{cases}$

Consider the first equation: $\frac{dx}{dt} = 3x - 2y$

$$\frac{dx}{dt} = 3x - 2k_2 e^{4t}$$

$x_h(t) = k_1 e^{3t}$

Try: $x_p(t) = A e^{4t}$

Plug in:

$$4A\cancel{e^{4t}} = 3A\cancel{e^{4t}} - 2k_2 \cancel{e^{4t}}$$

Solve for A: $4A = 3A - 2k_2$

$$A = -2k_2$$

So:

$$\boxed{x(t) = k_1 e^{3t} - 2k_2 e^{4t}}$$

Solution to the system:

$$\begin{cases} x(t) = k_1 e^{3t} - 2k_2 e^{4t} \\ y(t) = k_2 e^{4t} \end{cases}$$

3. $\begin{cases} \frac{dx}{dt} = -2x \longrightarrow & x(t) = k_1 e^{-2t} \\ \frac{dy}{dt} = x^2 - 4y \end{cases}$

$$\frac{dy}{dt} = (k_1 e^{-2t})^2 - 4y$$

$$\frac{dy}{dt} = -4y + k_1^2 e^{-4t} \longrightarrow \text{guess: } y_p(t) = Ate^{-4t}$$

$y_h(t) = k_2 e^{-4t}$

so $\frac{dy_p}{dt} = Ae^{-4t} - 4Ate^{-4t}$

$y_h(t) = k_2 e^{-4t}$

so $\frac{dy_p}{dt} = Ae^{-4t} - 4Ate^{-4t}$

plug in:

$Ae^{-4t} - \cancel{4Ate^{-4t}} = -\cancel{4Ate^{-4t}} + k_1^2 e^{-4t}$

$Ae^{-4t} = k_1^2 e^{-4t}$

so: $A = k_1^2$

general solution to the system

$$\boxed{\begin{aligned} x(t) &= k_1 e^{-2t} \\ y(t) &= k_2 e^{-4t} + k_1^2 t e^{-4t} \end{aligned}}$$

initial conditions: $x(0) = 3$, $y(0) = 5$

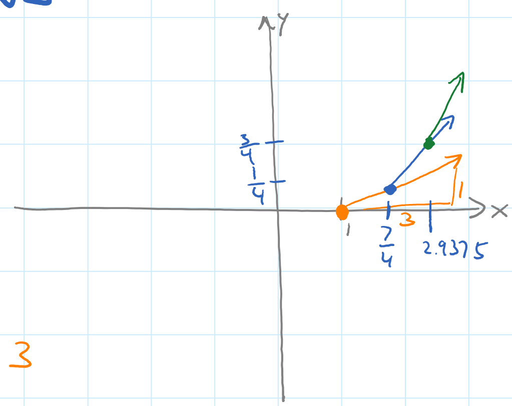
$x(0) = k_1 e^0 = 3$ so $k_1 = 3$

$y(0) = k_2 e^0 + \cancel{k_1^2 (0) e^0} = 5$ so $k_2 = 5$

particular solution: $\begin{cases} x(t) = 3e^{-2t} \\ y(t) = 5e^{-4t} + 9te^{-4t} \end{cases}$

APPROXIMATING SOLUTIONS

$$\begin{cases} \frac{dx}{dt} = 3x - 2y & \leftarrow f(x,y) \\ \frac{dy}{dt} = x + y & \leftarrow g(x,y) \end{cases}$$



At $(x,y) = (1,0)$: $\frac{dx}{dt} = 3(1) - 2(0) = 3$

$\begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases}$ $\frac{dy}{dt} = 1 + 0 = 1$

Let $\Delta t = \frac{1}{4}$. Then: $\begin{cases} x_1 = x_0 + f(x_0, y_0)\Delta t = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4} \\ y_1 = y_0 + g(x_0, y_0)\Delta t = 0 + 1\left(\frac{1}{4}\right) = \frac{1}{4} \end{cases}$

$\begin{cases} x_2 = x_1 + f(x_1, y_1)\Delta t = \frac{7}{4} + \left(3\left(\frac{7}{4}\right) - 2\left(\frac{1}{4}\right)\right)\frac{1}{4} \simeq 2.9375 \\ y_2 = y_1 + g(x_1, y_1)\Delta t = \frac{1}{4} + \left(\frac{7}{4} + \frac{1}{4}\right)\frac{1}{4} = \frac{3}{4} \end{cases}$

Define vector field $F(x, y)$:

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In[∘]:= F[x_, y_] := {3 x - 2 y, x + y}
```

Draw the phase portrait:

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In[∘]:= phaseplot = StreamPlot[F[x, y], {x, -4, 10}, {y, -4, 10}]
```
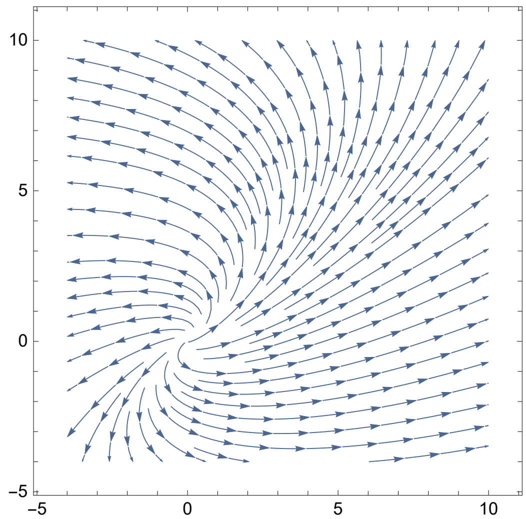

Out[∘]=



If $x(0) = 1$ and $y(0) = 0$, approximate $(x(1), y(1))$ using step size of 0.1:

```
In[∘]:= Y[0] = {1, 0};
Δt = 0.1;
Do[ Y[n + 1] = Y[n] + F[ First[Y[n]], Last[Y[n]] ] * Δt, {n, 0, 9}]
```

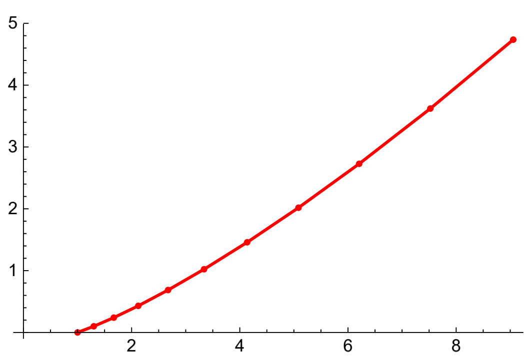
Print the table of approximation points:

```
In[∘]:= approx = Table[Y[n], {n, 0 , 10}]
Out[∘]= {{1, 0}, {1.3, 0.1}, {1.67, 0.24}, {2.123, 0.431}, {2.6737, 0.6864},
 {3.33853, 1.02241}, {4.13561, 1.4585}, {5.08459, 2.01792},
 {6.20638, 2.72817}, {7.52266, 3.62162}, {9.05514, 4.73605}}
```

Plot the approximate solution:

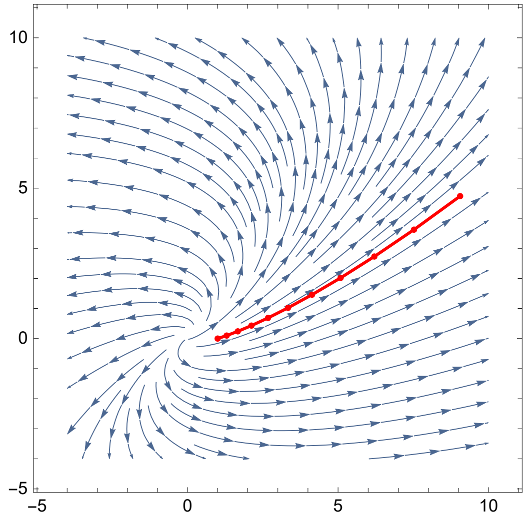
```
In[∘]:= approxPlot = ListPlot[approx, Joined → True, Mesh → All, PlotStyle → Red]
```

Out[∘]=



In[•]:=
```
Show[phaseplot, approxPlot]
```

Out[•]=



In[•]:=
```
StreamPlot[{v, -5 v - 6 y}, {y, -10, 10}, {v, -10, 10}]
```

Out[•]=