FIRST-ORDER LINEAR DIFFERENTIAL EQUATION:

a differential equation that can be put into the form

$$\frac{dy}{dt} = \underbrace{a(t)\,y + b(t)}_{\text{linear in } y}$$

$a(t)$, $b(t)$ are some functions of t

EXAMPLES: $\frac{dy}{dt} = 5y + 3t$ ← Here, $b(t) \neq 0$, so the equation is NONHOMOGENEOUS.

$\frac{dy}{dt} = e^{t+2}\,y$ ← Here, $b(t) = 0$, so the equation is HOMOGENEOUS.

LET'S SOLVE $\frac{dy}{dt} = 5y + 3t$

(a) Associated homogeneous equation: $\frac{dy}{dt} = 5y$

Solution: $y_h(t) = e^{5t}$

General solution to this eq: $k \cdot y_h(t) = ke^{5t}$

(b) Find any particular solution $y_p(t)$ to $\frac{dy}{dt} = 5y + 3t$

Want $y_p(t)$ whose derivative contains terms like $y_p(t)$ and $3t$

Method of Undetermined Coefficients

guess: $y_p(t) = At + B$

so $\frac{dy_p}{dt} = A$ plug in: $\frac{dy_p}{dt} = 5y_p + 3t$

$A = 5(At + B) + 3t$

$A = 5B + (5A + 3)t$

$$\boxed{y_p(t) = \frac{-3}{5}t - \frac{3}{25}}$$

$$\boxed{y_p(t) = -\frac{3}{5}t - \frac{3}{25}}$$

$A = 5B + (5A+3)t$

$A = 5B \qquad 5A+3=0$

$B = \frac{A}{5} = \frac{-3}{25} \qquad A = -\frac{3}{5}$

(c) General solution to $\frac{dy}{dt} = 3y + 5t$ is $k \cdot y_h(t) + y_p(t)$

follows from the Extended Linearity Principle

$$\boxed{y(t) = k \cdot e^{5t} - \frac{3}{5}t - \frac{3}{25}}$$

diff

(d) Check: $\frac{dy}{dt} = 5y + 3t$

$$5ke^{5t} - \frac{3}{5} = 5\left(ke^{5t} - \frac{3}{5}t - \frac{3}{25}\right) + \cancel{3t}$$

$\cancel{-3t}$

✓

PRACTICE PROBLEMS

1. $\frac{dy}{dt} = 3y + e^{-t}$

(a) $y_h(t) = e^{3t}$

(b) guess: $y_p(t) = Ae^{-t}$

plug in: $-A\cancel{e^{-t}} = 3A\cancel{e^{-t}} + \cancel{e^{-t}}$

$-A = 3A + 1$

$-4A = 1$ so $A = -\frac{1}{4}$

$$\boxed{\text{General solution: } y(t) = ke^{3t} - \frac{1}{4}e^{-t}}$$

2. $\frac{dy}{dt} = -y + \sin(t)$

(a) $y_h(t) = e^{-t}$

(b) guess: $y_p(t) = A\cos(t) + B\sin(t)$

plug in: $-A\sin(t) + B\cos(t) = -\left(A\cos(t) + B\sin(t)\right) + \sin(t)$

$-A = -B + 1$ and $B = -A$

$A = -\frac{1}{2}$ and $B = \frac{1}{2}$

(c) general solution: $y(t) = ke^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$

First-Order Linear Differential Equations

Math 230

Solve the linear differential equations by completing the following steps:

(a) Write the associated homogeneous equation and find its general solution $ky_h(t)$.

(b) Find *any* particular solution $y_p(t)$ to the nonhomogeneous equation.

(c) The general solution to the nonhomogeneous equation is $ky_h(t) + y_p(t)$.

(d) Check your solution by plugging it back in to the nonhomogeneous equation.

1. $\dfrac{dy}{dt} = 3y + e^{-t}$

The associated homogeneous equation $\frac{dy}{dt} = 3y$ has solution $y_h(t) = e^{3t}$.

For the particular solution, guess $y_p(t) = Ae^{-t}$. Substitute into the differential equation and solve to find $A = -\frac{1}{4}$.

Thus, the general solution is: $y(t) = ke^{3t} - \frac{1}{4}e^{-t}$

2. $\dfrac{dy}{dt} = -y + \sin(t)$

Associated homogeneous equation: $\frac{dy}{dt} = -y$ has solution $y_h(t) = e^{-t}$.

Particular solution: guess $y_p(t) = A\sin t + B\cos t$; substitute into the differential equation and solve to find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Thus, the general solution is: $y(t) = ke^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$

3. $\dfrac{dy}{dt} = y + e^{t}$

Associated homogeneous equation: $\frac{dy}{dt} = y$ has solution $y_h(t) = e^{t}$.

Particular solution: we would like to guess e^t, but this is already y_h, so we instead choose $y_p(t) = Ate^t$. Substitute into the differential equation and solve to find $A = 1$.

Thus, the general solution is: $y(t) = ke^t + te^t$

4. $\dfrac{dy}{dt} = y + t + e^{2t}$

Associated homogeneous equation: $\frac{dy}{dt} = y$ has solution $y_h(t) = e^{t}$.

Particular solution: guess $y_p(t) = A + Bt + Ce^{2t}$; substitute into the differential equation and solve to find $A = -1$, $B = -1$, and $C = 1$.

Thus, the general solution is: $y(t) = ke^t - 1 - e^t + e^{2t}$