Math 230, 4 September 2020

AUTONOMOUS EQUATIONS WITH PARAMETERS

EXAMPLE: $\frac{dy}{dt} = y^2 - 4y + k$ ↖parameter

- What are the equilibrium solutions?

$$0 = y^2 - 4y + k \quad \text{so} \quad y = \frac{4 \pm \sqrt{4^2 - 4k}}{2} = 2 \pm \sqrt{4-k}$$

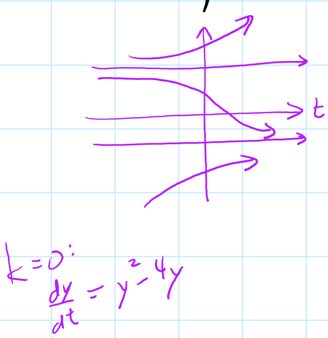
$$\sqrt{4^2-4k} = \sqrt{4(4-k)} = 2\sqrt{4-k}$$

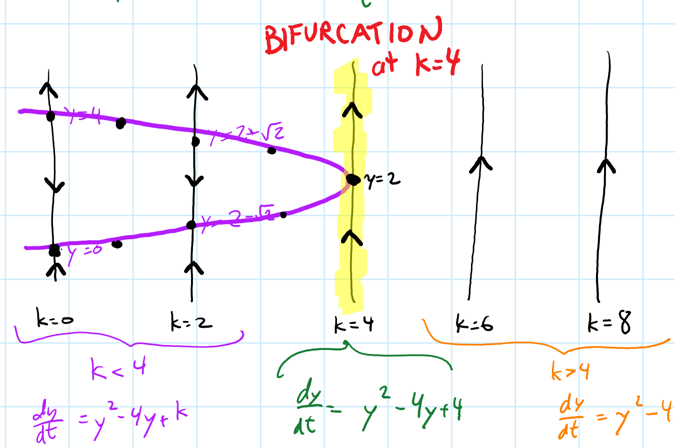
If $4-k>0$, or $4>k$, then 2 equilibrium solutions: $y = 2 \pm \sqrt{4-k}$.

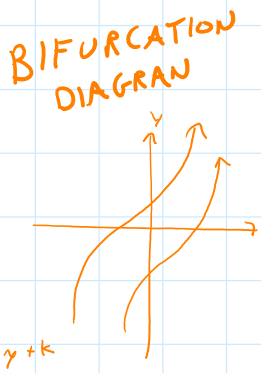
If $4-k=0$, or $4=k$, then 1 equilibrium solution: $y=2$

If $4-k<0$, or $4<k$, then no equilibrium solutions

- Plot some phase lines:







- Describe the bifurcation:

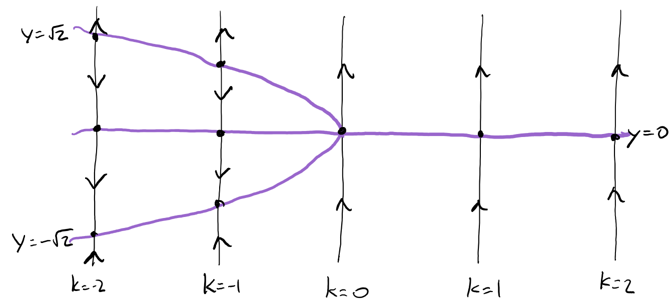
As k goes from less than 4 to greater than 4, the two equilibrium points (a source and a sink) get closer together, then merge to form a node, which disappears.

Worksheet Solutions
Math 230, 4 September 2020

1. $\frac{dy}{dt} = y^4 + ky^2$

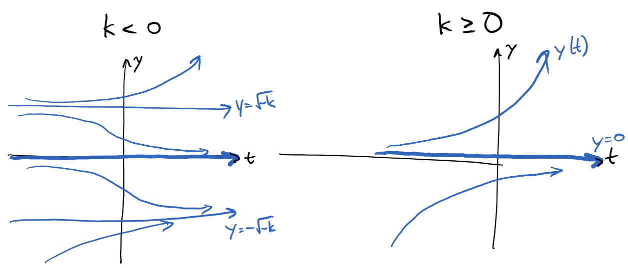
- If $k < 0$, then equilibrium solutions $y=0$, $y = \pm\sqrt{-k}$
- If $k \geq 0$, then the only equilibrium solution is $y=0$.

Bifurcation occurs at $k=0$, since the number of equilibrium solutions changes.



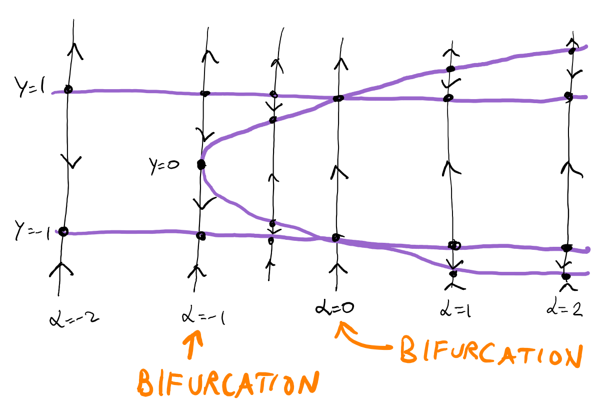
Describe the bifurcation: As k goes from negative to positive, three equilibrium solutions (sink, source, and node) merge into a single node

Solution sketch:



2. $\frac{dy}{dt} = (y^2 - \alpha - 1)(y^2 - 1)$

Equilibrium solutions: $y = \pm 1$ and $\underbrace{y = \pm\sqrt{\alpha+1}}_{\text{exists only if } \alpha \geq -1}$



As α increases through $\alpha=-1$, a node appears, then splits into a source and a sink.

As α increases through $\alpha = 0$, the source and sink pass through nodes at $y=-1$ and $y=1$. The source becomes a sink, and the sink becomes a source.

3. Here is one possible answer: