EULER'S METHOD

$$\frac{dy}{dt} = 2y + \sin(t)$$

Questions:

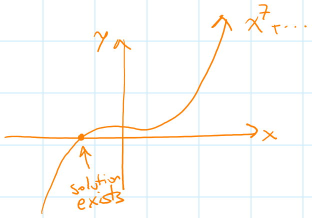
1. Does the approximation converge as the step size decreases?
2. If so, does it converge to the exact solution of the diff. eq.?

EXISTENCE AND UNIQUENESS

Comparison: $x^7 - 5x^3 + 2x - 3 = 0$

Is there a solution?

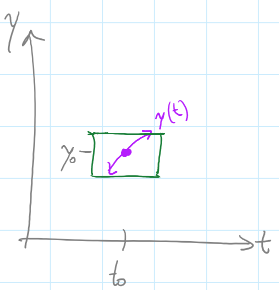
Is the solution unique?



$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

EXISTENCE: If $f(t,y)$ is continuous in a rectangle containing (t_0, y_0), then a solution exists through this point.

UNIQUENESS: If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are both continuous in a rectangle containing (t_0, y_0), then the solution through (t_0, y_0) is unique.



1. Consider the differential equation $\cos(t)\,y' - \sin(t)\,y = 3t\cos(t)$. ($y' = \frac{dy}{dt}$)

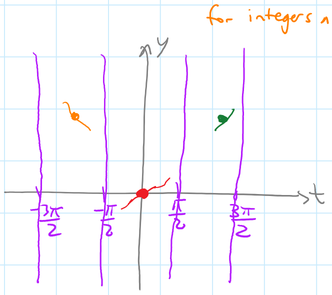
(a) At what points (t, y) does a solution exist?

Solve for $\frac{dy}{dt}$: $\quad \frac{dy}{dt} = \frac{3t\cos(t) + \sin(t)y}{\cos(t)}$ or $\frac{dy}{dt} = 3t + \tan(t)y$ ← $f(t,y)$

Where is $f(t,y) = 3t + \tan(t)\,y$

not continuous at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ $(n+\frac{1}{2})\pi$ for integers n

At (t,y) with $t \neq (n+\frac{1}{2})\pi$ (for integer n), there exists a solution.



(b) At what points is the solution unique?

$\frac{\partial f}{\partial y} = \tan(t)$

continuous for $t \neq (n+\frac{1}{2})\pi$, n integer

At (t,y) with $t \neq (n+\frac{1}{2})\pi$, the solution is unique.

(c) If a solution $y(t)$ is such that $y(0) = 0$, what is the largest interval on which we can guarantee that this solution is unique?

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Compute an approximation with 10 steps:

In[18]:=
```
Clear[y]
f[t_, y_] := 2 y - Sin[t]
y[0] = 3;
steps = 10;
Δt = 1.0/steps;
Do[y[n + 1] = f[Δt * n, y[n]] * Δt + y[n], {n, 0, steps - 1}]
points10 = Table[{Δt * n, y[n]}, {n, 0, steps}]
```

In[25]:=
```
y[10]
```

Out[25]= 17.8218

In[26]:=
```
points10 = Table[{Δt * n, y[n]}, {n, 0, steps}]
```

Out[26]= {{0., 3}, {0.1, 3.6}, {0.2, 4.31002}, {0.3, 5.15215}, {0.4, 6.15303}, {0.5, 7.3447}, {0.6, 8.76569}, {0.7, 10.4624}, {0.8, 12.4904}, {0.9, 14.9168}, {1., 17.8218}}

Compute an approximation with 100 steps:

In[27]:=
```
Clear[y]
f[t_, y_] := 2 y - Sin[t]
y[0] = 3;
steps = 100;
Δt = 1.0/steps;
Do[y[n + 1] = f[Δt * n, y[n]] * Δt + y[n], {n, 0, steps - 1}]
points100 = Table[{Δt * n, y[n]}, {n, 0, steps}]
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Plot both approximation together:

In[17]:=
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ListPlot[{points10, points100}]
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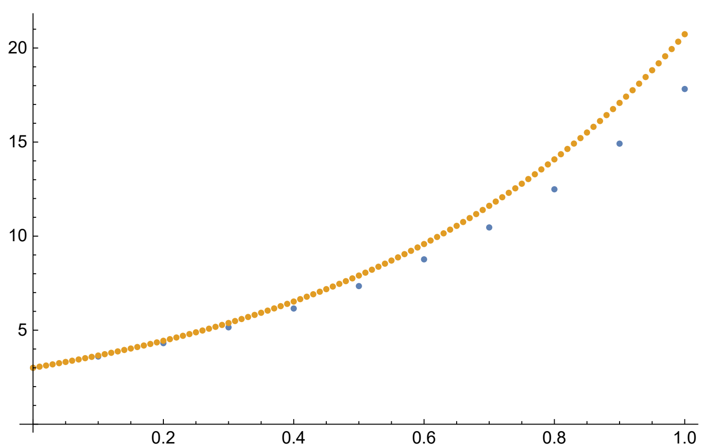
Out[17]=



Existence and Uniqueness

Math 230

1. Suppose $f(t, y)$ and $\frac{\partial y}{\partial t}$ are continuous for all (t, y). Also suppose that $y_1(t) = 3$, $y_2(t) = 6$, and $y_3(t) = t^2 + 8$ are solutions to $\frac{dy}{dt} = f(t, y)$ for all t.

   (a) If a particular solution satisfies $y(0) = 4$, explain why $3 < y(t) < 6$ for all t for this solution.

   Since there is a unique solution at any point (t, y), the equilibrium solutions $y = 3$ and $y = 6$ are the only solutions at those y-values. That is, no other solution may cross the horizontal lines $y = 3$ and $y = 6$. Thus, $3 < y(t) < 6$ for all t.

   (b) What lower and upper bounds can you give for a particular solution that satisfies $y(0) = 7$?

   $6 < y(t) < t^2 + 8$ for all t

   (c) What lower and upper bounds can you give for a particular solution that satisfies $y(0) = 9$?

   $t^2 + 8 < y(t)$ for all t

2. Consider the autonomous differential equation $\frac{dy}{dt} = |y|$.

   (a) What are the equilibrium solutions?

   The only equilibrium solution is $y = 0$.

   (b) For what values of y does a solution exist?

   Since $f(t, y) = |y|$ is continuous everywhere, a solution exists at any point (t, y).

   (c) For what values of y is there a unique solution?

$$\frac{\partial f}{\partial y} = \begin{cases} -1 & \text{if } y < 0 \\ 1 & \text{if } y > 0 \end{cases}$$

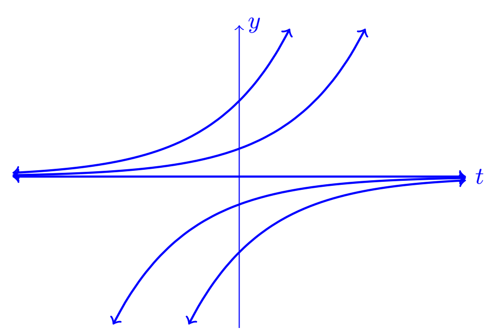
   Since $\frac{\partial f}{\partial y}$ is continuous when $y \neq 0$, the uniqueness theorem guarantees that the solution is unique wherever $y \neq 0$.

   (d) Find all solutions, and sketch the family of solutions. *Hint*: Consider the cases $y > 0$ and $y < 0$ separately, and separate variables. Then consider the case $y = 0$.

   If $y > 0$, then $\frac{dy}{dt} = y$, and we can separate variables to obtain $y = Ke^t$ for some constant $K > 0$.

   If $y < 0$, then $|y| = -y$, so $\frac{dy}{dt} = -y$. Separating variables gives $\frac{dy}{y} = -dt$, and we integrate to obtain $\ln|y| = -t + C$. Since y is negative, $\ln|y| = \ln(-y)$, and we obtain $y = Ke^{-t}$ for some constant $K < 0$.

   Together with the equilibrium solution $y = 0$, we the family of solutions looks like this:



3. Consider the autonomous differential equation $\frac{dy}{dt} = \frac{1}{(1+y)^2}$.

(a) For what values of y is there a unique solution?

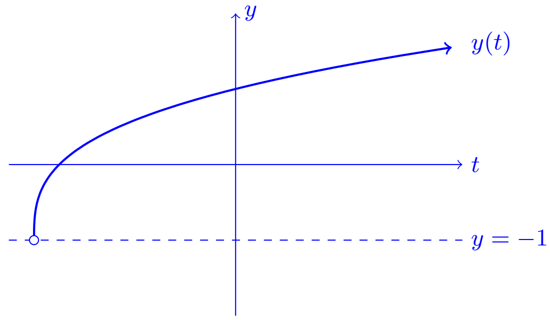
Since $f(t, y) = (1 + y)^{-2}$ and $\frac{\partial f}{\partial y} = -2(1 + y)^{-3}$ are continuous whenever $y \neq -1$, a unique solution exists at any point (t, y) with $y \neq -1$.

(b) Find all solutions. *Hint*: Are there any equilibrium solutions? Now separate variables!

There are no equilibrium solutions. By separating variables, we find that the general solution is $y(t) = \sqrt[3]{3t + C} - 1$ for some constant C.

(c) Find the particular solution $y(t)$ such that $y(0) = 1$. What is the largest interval of t-values on which this solution exists? Sketch the solution.

The particular solution is $y(t) = \sqrt[3]{3t + 8} - 1$, which exists on the interval $\left(\frac{-8}{3}, \infty\right)$. A sketch of this solution is:



4. Consider the autonomous differential equation $\frac{dy}{dt} = 1 + y^2$.

(a) For what values of y is there a unique solution?

Since $f(t, y) = 1 + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous everywhere, a unique solution exists at any point (t, y).

(b) Find all solutions.

There are no equilibrium solutions. By separating variables, we find that the general solution is $y = \tan(t + C)$ for some constant C.

(c) Find the particular solution $y(t)$ such that $y(0) = 0$. What is the largest interval on which this solution exists? *Hint*: What is the domain of the solution?

The particular solution is $y(t) = \tan(t)$, and the largest interval on which this solution is defined is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.