Study differential equations — 3 approaches

Analytic: finding a formula of a solution

Qualitative: understanding properties of the solution, without a formula

Numerical: using a computer to calculate points on the solution

SEPARABLE EQUATIONS: $\frac{dy}{dt} = g(t)\,h(y)$

We can rewrite: $\frac{dy}{h(y)} = g(t)\,dt$ ← "separated" the t's and y's

If we can integrate both sides, and if we can solve the resulting equation for $y(t)$, then we have a solution to the diff eq.

EXAMPLE: $\boxed{\underbrace{\frac{dy}{dt} = \frac{-t}{y}}_{\text{diff. eq.}} \quad \text{and} \quad \underbrace{y(0) = 2}_{\text{initial value}}}$ initial-value problem

Separate variables: $\int y\,dy = \int -t\,dt$

integrate: $\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$

solve for $y(t)$: $y^2 = -t^2 + C_1$ where $C_1 = 2C$

$y(t) = \pm\sqrt{C_1 - t^2}$ ← general solution

initial value: $y(0) = 2 \Rightarrow y(0) = \pm\sqrt{C_1 - 0^2} = \pm\sqrt{C_1}$

$2 = \pm\sqrt{C_1}$ so $2 = \sqrt{C_1}$ and $C_1 = 4$

particular solution: $\boxed{y(t) = \sqrt{4 - t^2}}$

Sometimes, we need to compute integrals such as:

$$\int \frac{1}{y(y+2)}\,dy$$

→ use partial fractions: $\frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2}$

Solve for A and B. We'll say more about this on Wednesday.
You can also use technology such as Wolfram Alpha.

SOLUTIONS TO SEPARATION OF VARIABLES PROBLEMS

1. $\dfrac{dy}{dt} = \dfrac{15-y}{20}$

These steps apply if $y \neq 15$.

$$\int \frac{dy}{15-y} = \int \frac{dt}{20}$$

$y \neq 15$

$$-\ln|15-y| = \frac{t}{20} + C$$

$$\ln|15-y| = C - \frac{t}{20}$$

$$|15-y| = e^{C - t/20} = K e^{-t/20} \qquad (K = e^C)$$

$$15 - y = \pm K e^{-t/20}$$

$$y(t) = 15 - K e^{-t/20}$$

(a) $y(0) = 20 \Rightarrow y(t) = 15 + 5e^{-t/20}$

(b) $y(0) = 15 \Rightarrow \underbrace{y(t) = 15}_{\text{equilibrium solution}}$

2. $\dfrac{dy}{dt} = y^2 t \Rightarrow \displaystyle\int \frac{dy}{y^2} = \int t\, dt$

This step requires that $y \neq 0$.

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

$$\frac{1}{y} = -\frac{1}{2}t^2 + C$$

$$y = \frac{1}{C - \frac{1}{2}t^2} \quad \text{or} \quad \boxed{y(t) = \frac{2}{C - t^2}}$$

missing equilibrium solution: $\boxed{y = 0}$

3. $\underbrace{\dfrac{dy}{dt} = y(y+2)}$

"autonomous" equation — right side involves only y, not t

$$\int \frac{dy}{y(y+2)} = \int dt$$

PARTIAL FRACTIONS: $\dfrac{1}{y(y+2)} = \dfrac{A}{y} + \dfrac{B}{y+2}$

$$1 = A(y+2) + By$$

If $y=0$: $\quad 1 = A(2) + 0 \implies A = \frac{1}{2}$

If $y=-2$: $\quad 1 = A(0) + B(-2) \implies B = -\frac{1}{2}$

$$\int \left(\frac{1}{2y} - \frac{1}{2(y+2)} \right) dy = \int dt$$

$$\frac{1}{2}\ln|y| - \frac{1}{2}\ln|y+2| = t + C$$

$$\ln|y| - \ln|y+2| = 2t + C$$

$$\ln\left|\frac{y}{y+2}\right| = 2t + C$$

$$\left|\frac{y}{y+2}\right| = e^{2t+C}$$

$$\frac{y}{y+2} = ke^{2t}$$

$$y = (y+2)ke^{2t}$$

$$y(1-ke^{2t}) = 2ke^{2t}$$

$$\boxed{y(t) = \frac{2ke^{2t}}{1-ke^{2t}}}$$

4. MIXING PROBLEM — This is a more involved problem, given for extra practice.

(a) quantities: t = time (days)

$f(t)$ = amount of pollution in lake at time t

$L = 400{,}000 \text{ m}^3$ = volume of the lake

$1000 \text{ m}^3/\text{day}$ = flow into/out of lake

(b) Equation: $\frac{df}{dt} = (\text{rate in}) - (\text{rate out})$

rate in: 7 g. of pollution per day

rate out: $\left(\frac{f(t)}{400{,}000}\ \frac{\text{g}}{\text{m}^3}\right)\left(1000\ \frac{\text{m}^3}{\text{day}}\right) = \frac{f(t)}{400}\ \frac{\text{g}}{\text{day}}$

$$\frac{df}{dt} = 7 - \frac{f(t)}{400} = \frac{2800 - f(t)}{400}$$

(c) Equilibrium solution: $f(t) = 2800$

(d) Solve by separation of variables:

$$\frac{df}{2800 - f(t)} = \frac{dt}{400}$$

$$\frac{df}{2800 - f(t)} = \frac{dt}{400}$$

$$-\ln|2800 - f(t)| = \frac{t}{400} + C$$

$$\boxed{f(t) = 2800 - Ke^{-t/400}}$$

$$2800 - f(t) = Ke^{-t/400}$$

(e) $f(0) = 0$ implies $\boxed{f(t) = 2800 - 2800\,e^{-t/400}}$

(f) As $t \to \infty$, $f(t) \to 2800$.

The concentration approaches $\dfrac{2800 \text{ g}}{400000 \text{ m}^3} = \dfrac{2.8 \text{ g}}{400 \text{ m}^3} = 7 \text{ mg}/\text{m}^3$