Example: Population modeling. *The growth rate of a population is proportional to the size of the population.*

Assumption

(a) Based on the assumption, what quantities are necessary to model the population?

time: t hours? years?

population size: $P(t)$ people? cells? individuals

note: $\frac{dP}{dt}$ is the growth rate

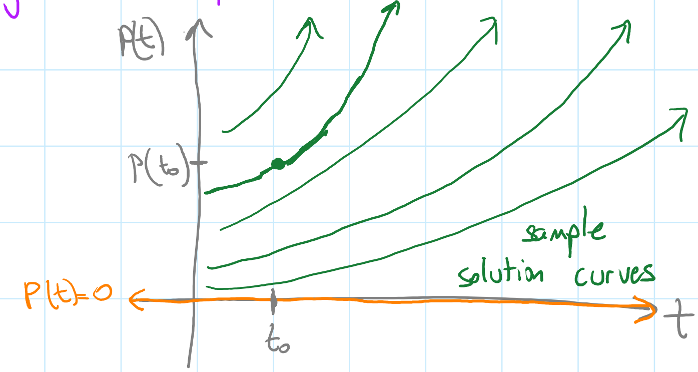
constant of proportionality k

(b) Using notation from part (a), convert the assumption to a differential equation.

$$\boxed{\frac{dP}{dt} = k\,P(t)}$$

(c) Without solving the differential equation, can we sketch the general shape of solutions?

Assume $k > 0$.



If $P(t_0) > 0$, for some time t_0,
then $\frac{dP}{dt}(t_0) = k\,P(t_0) > 0$,
so the population grows, and grows, and grows, ...

If $P(t_0) = 0$, then $\frac{dP}{dt}(t_0) = 0$, so the population never grows.

$P(t) = 0$ is a solution — an equilibrium solution

(d) What additional assumptions, if any, did we make?

$k > 0$

2. **Spread of a rumor.** *In a city with a fixed population of P persons, the rate of change of the number of people who have heard a certain rumor is proportional to the number who have not heard the rumor.*

Quantities: time t

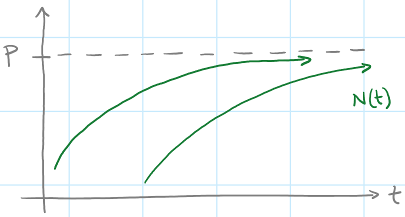
population size P

number of people who have heard the rumor at time t: $N(t)$

constant of proportionality k

Differential Equation: $\frac{dN}{dt} = k(P - N)$

Sketch of Solutions:



Assumptions: $k > 0$

3. **Population model with a carrying capacity.** *If a population size is small, then its rate of growth is proportional to its size. However, if the population is larger than some fixed carrying capacity, then its growth rate is negative.*

Quantities: time: t

population size: $P(t)$

constant of proportionality: k

fixed carrying capacity: C

Differential Equation: we want $\frac{dP}{dt} \approx kP$ if P is small

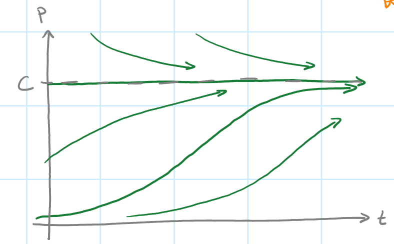
and $\frac{dP}{dt} \to 0$ as $P \to C$

Try: $\frac{dP}{dt} = k \cdot (\text{something}) \cdot P$

close to 1 if P is small, and approaches 0 as $P \to C$

For example: $\frac{dP}{dt} = k\left(1 - \frac{P}{C}\right)P$ Logistic Population Model

Sketch of Solutions:



Note: equilibrium solutions $P(t) = C$ and $P(t) = 0$

4. Deer population. *A population of deer live in an area with a carrying capacity of 10,000 deer. If the population is small, then the population grows in proportion to its size. If the population is larger than the carrying capacity, then its growth rate is negative. In addition, one tenth of the deer are taken by hunters each year.*

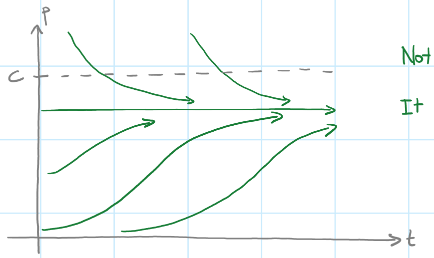
Quantities: t, $P(t)$, C, and k as before

We will assume that t is measured in years.

Differential Equation: $\frac{dP}{dt} = k\left(1 - \frac{P}{C}\right)P - \frac{P}{10}$

This term accounts for one-tenth of the deer being removed from the population each year.

Sketch of Solutions:



Note that if $P(t) = C$, then $\frac{dP}{dt} < 0$.

It seems there should be an equilibrium solution $P(t) = r$, for some $r < C$.