Final Exam Guidance

MATH 220 E • Spring 2025

The following is intended to help you focus your studying for the final exam. The final exam is cumulative, since linear algebra is cumulative — for example, you won't understand Chapter 6 in the text if you don't know the concepts from Chapters 1–5. The exam will emphasize material from later in the semester (from Chapters 5–8), but it's necessary to know all the material we have studied as thoroughly as possible.

The following list states things you should know. I don't claim for this list to be exhaustive, but it should help you organize your studying. If you have questions about this list or about any topics, *now* is the time to formulate and ask those questions! I want to help.

Things You Should Know

Vectors and matrices

- How to add vectors and multiply vectors by scalars
- How to add and multiply matrices
- Matrix terms such as: upper-triangular, diagonal, transpose, identity, symmetric
- How to calculate dot products and the connection between dot products and matrix multiplication
- What does it mean for a matrix to be invertible, and how to find the inverse of a matrix

Systems of Equations

- How to solve a system of linear equations
- Notation: $A\mathbf{x} = \mathbf{b}$
- Terminology such as: coefficient matrix, augmented matrix, free variable, pivot, solution set, vector form

Linear combinations, span, and linear independence

- Definition of linear combination, and how to show a vector is or is not a linear combination of a given set of vectors
- Definition of span (as a noun), and how to show that a vector is or is not within the span of a given set of vectors
- Definition of span (as a verb), and how to show that a list of vectors spans a given space
- Definition of linear independence, and how to determine whether or not a list of vectors is linearly independent
- If vectors are linearly dependent, how to find a dependence relation
- Understand how these concepts apply to all vector spaces, not just $\mathbb{R}^n$

Subspaces, basis, and dimension

- Definition of subspace, and how to show that a given set of vectors is or is not a subspace using the definition
- Definitions of basis and dimension, and how to find the basis and dimension of a given subspace (in all vector spaces, not just $\mathbb{R}^n$)
- Row space, column space, and null space of a matrix, and how to find a basis for each
- Rank and nullity of a matrix, and how these are related
- In general, how do you write a subspace as a span? How do you find a basis for a subspace?

Linear transformations

- Definition of linear transformation, and how to show that a given function is or is not a linear transformation
- Kernel and range of a linear transformation, how to find these, and how they relate to the null space and column space of a matrix
- Given a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, how to find the matrix that performs the transformation

Determinants

- What is the determinant of a matrix, and how is it calculated
- What information does the determinant provide about a matrix, and about the linear transformation defined by multiplication by that matrix
- Properties of the determinant, especially Theorem 5.12
- How the determinant relates to the other statements in the Unifying Theorem

Eigenvectors, eigenvalues, and diagonalization

- What it means for a vector to be an eigenvector of a matrix
- What it means for a scalar λ to be an eigenvalue of a matrix
- How to find the eigenvalues of a matrix (use the characteristic polynomial)
- How to find eigenvectors of a matrix — you should know *why* the process works the way it does, not just *how* to do the process
- What is an eigenspace and how to find a basis for an eigenspace
- Algebraic multiplicity and geometric multiplicity: what these are, how to find them, and how they relate to each other
- How the concepts of eigenvalues/eigenvectors fit within the Unifying Theorem (i.e., with $\lambda = 0$)

- What it means for a matrix to be diagonalizable
- How diagonalization is useful for computing powers of a matrix
- How to determine whether or not a matrix is diagonalizable
- If a matrix is diagonalizable, how to find the matrices P and D that are part of the diagonalization
- You do *not* need to know about Markov chains for the exam

Orthogonality in $\mathbb{R}^n$

- Dot product: how is it defined, and what is it useful for
- Orthogonal complement: what is $S^\perp$, and how to find a basis for $S^\perp$
- Projection of a vector onto a vector: what this means, and how to compute it
- What is an orthogonal set (or basis) and how to check whether a set of vectors is orthogonal
- Projection of a vector onto a subspace: what this means, and how to compute it (*You MUST remember that you need an orthogonal basis for S in order to use the formula for the projection*)
- You do *not* need to know about least squares for the exam

Abstract vector spaces

- Definition of a vector space
- Examples of vector spaces other than $\mathbb{R}^n$
- Be able to do problems about linear combinations, spans, linear independence, subspace, bases, and dimension, using vector spaces other than $\mathbb{R}^n$

The Unifying Theorem

This ties (most of) the course together! You should be able to list many statements that are all equivalent to one another. Some of the statements are about vectors, some of the statements are about a matrix, and some of the statements are about a linear transformation. Of course, the matrix is *made out of* the vectors and the linear transformation is *performed by* the matrix.

What can I use on the exam?

- *You may use one* $8\frac{1}{2} \times 11$*-inch (one side) piece of paper of notes prepared in advance.*
- The exam will involve only minimal row-reduction and no tedious arithmetic. Calculators are *not necessary*. However, you may use a calculator, but only for row-reducing matrices or simple arithmetic, and you must state where exactly you used your calculator. You may not use a phone or computer.

How should I study?

First, understand that people learn differently and process information in different ways and at different speeds. I suggest:

- Read through the text again and think about whether or not the main theorems make intuitive sense. Can you explain them out loud?
- Do a few problems each day, ramping up as we get closer to the exam day. The supplementary exercises at the end of each chapter are great sources of practice problems. Talk with your classmates about the problems. Talk with me and visit the help sessions if you want to make sure things are correct or if you want to chat.
- Work on fluency! The exam is timed and you want to be able to do some of the problems efficiently. Perhaps give each other a few selected problems from a few different sections and time yourselves. This way, you won't know which section the problem came from.
- COME VISIT ME AND ASK QUESTIONS.

Good luck! You've learned a LOT this semester! You're going to do great on the exam!