Linear Algebra – Day 38

MATH 220

Not every topic in this course appears among these review problems. If you find you want practice with a specific topic and want suggestions, I am HAPPY to give them. Please just ask.

1. List *as many* items that could be part of the Unifying Theorem as you can.

2. Consider the vector space $\mathbb{R}^{4\times 4}$ consisting of all 4×4 matrices real number entries. Let W be the subset consisting of all symmetric matrices that have every diagonal entry equal to 0. Prove that W is a subspace of $\mathbb{R}^{4\times 4}$ and find a basis for W.

3. Find the matrix A such that the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ has the overall effect of rotating $\mathbf{x}$ by 90 degrees, then reflecting in the line $y = -x$, then doubling its length.

4. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\0\\0\\2\\1\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}1\\1\\2\\3\\4\end{bmatrix}$. Is the vector $\mathbf{v} = \begin{bmatrix}7\\4\\5\\11\\12\end{bmatrix}$ in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

5. Consider the vector space $\mathbf{P}^3$. Do the polynomials $f_1(x) = 4x^3 - 2x^2 + x$, $f_2(x) = x^2 - 1$, $f_3(x) = x^2 - x$, and $f_4(x) = 3$ span $\mathbf{P}^3$? Why or why not?

6. Consider the 4×5 matrix

$$A = \begin{bmatrix}1 & 0 & -1 & 3 & -1\\ 1 & 0 & 0 & 2 & -1\\ 2 & 0 & -1 & 5 & -1\\ 0 & 0 & -1 & 1 & 0\end{bmatrix}.$$

(a) What is rank(A)?

(b) Find bases for col(A), null(A), and row(A).

7. Let V be the subset of $\mathbf{P}^2$ consisting of polynomials $\mathbf{p}(x)$ with the property $\mathbf{p}(0) = 0$ and $\mathbf{p}(1) = 0$. Determine whether or not V is a subspace of $\mathbb{P}^2$. If it is not, explain why. If it is, find a basis for V.

8. Suppose that the augmented matrix for a system of linear equations is

$$\left[\begin{array}{ccc|c}0 & 1 & 4 & \frac{k}{2}\\ 1 & 0 & 8 & 1\\ 1 & -1 & k^2 & 0\end{array}\right]$$

(a) For which values of k, if any, does the system have no solutions?

(b) For which values of k, if any, does the system have infinitely many solutions?

(c) For which values of k, if any, does the system have a unique solution?

9. For each of the following, the answer is *always*, *sometimes*, or *never* true.

(a) If a 3×4 matrix has three pivots after row reduction, then its columns form a linearly independent set.

(b) The span of three (or more) vectors in $\mathbb{R}^3$ is all of $\mathbb{R}^3$.

(c) If the reduced echelon form of A is I_n, then A is invertible.

(d) Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. If $\mathbf{c}$ represents a different vector in $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{c}$ has a unique solution.

10. Consider the following vectors in $\mathbb{R}^3$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) The vector $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$ is in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Write $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$? Explain.

(c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly independent set or a linearly dependent set? Explain.

11. Suppose $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation with matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) What are the domain and codomain of T? (Make sure to specify which is which.)

(b) Is T one-to-one? Is T onto? Explain.

12. Consider the following matrix and its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 10 & 28 & 6 \\ 2 & 6 & 1 & -1 & 17 \\ 3 & 9 & -8 & -30 & 28 \\ -1 & -3 & 1 & 5 & 41 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of $\text{col}(A)$, $\text{row}(A)$, and $\text{null}(A)$.

(b) What is the rank of A?

13. Let $S = \{1 - x + x^2, 2 + x - x^2, -1 - 5x + 5x^2, x\}$ be a set of polynomials in $\mathbf{P}^2$.

(a) Is S linearly independent or dependent? Explain.

(b) Does S span $\mathbf{P}^2$? Explain.

(c) Does S form a basis for $\mathbf{P}^2$? Why or why not?

14. Let W be the subset of all matrices A in $\mathbb{R}^{2\times 2}$ whose determinant is equal to 0. Determine whether or not W is a subspace of $\mathbb{R}^{2\times 2}$.

15. Recall that if A is a matrix, then the *transpose* of A, denoted A^T, is the matrix whose rows are the columns of A and whose columns are the rows of A (in order).

Let V be the subspace of $\mathbb{R}^{2\times 2}$ consisting of all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with the property

$$A = -A^T.$$

(a) Without finding a basis for V or (directly or indirectly) discussing free variables, explain why $0 < \dim(V) < 4$.

(b) Find a basis for V.

16. Find a basis for $S^\perp$ if $S = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

17. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find the eigenvalues of A, a basis for each eigenspace, and if possible, diagonalize A.