Review Problems

Linear Algebra – Day 38

MATH 220

Not every topic in this course appears among these review problems. If you find you want practice with a specific topic and want suggestions, I am HAPPY to give them. Please just ask.

1. List *as many* items that could be part of the Unifying Theorem as you can.

Let S be a set of n vectors in $\mathbb{R}^n$, A a matrix whose columns are the vectors in S, and $T:\mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$.

Then all of the statements at right are equivalent (all true or all false)

- S spans $\mathbb{R}^n$
- S is linearly independent
- S is a basis for $\mathbb{R}^n$
- A is invertible
- rref (A) is the identity matrix
- $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$
- col $(A) = \mathbb{R}^n$
- row $(A) = \mathbb{R}^n$
- rank$(A) = n$
- $\det(A) \neq 0$
- T is onto
- T is one-to-one
- $\ker(T) = \{\vec{0}\}$
- $\lambda = 0$ is not an eigenvalue of A

2. Consider the vector space $\mathbb{R}^{3\times 3}$ consisting of all 3×3 matrices real number entries. Let W be the subset consisting of all symmetric matrices that have every diagonal entry equal to 0. Prove that W is a subspace of $\mathbb{R}^{3\times 3}$ and find a basis for W.

Matrices in W have the form $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$.

[This is the 3x3 case. The 4x4 case is similar.]

Let $S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$, and then $W = \text{span}(S)$

Since W can be written as a span, W is a subspace

The set S is linearly independent and spans W, so S is a basis for W.

3. Find the matrix A such that the matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ has the overall effect of rotating $\mathbf{x}$ by 90 degrees, then reflecting across the line $y = -x$, then doubling its length.

This matrix is given by:

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{\text{matrix for doubling length}} \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{\text{matrix for reflecting across the line } y=-x} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{matrix for 90 degree rotation (counterclockwise)}} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\2\\1\\1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\0\\0\\2\\1\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}1\\1\\2\\3\\4\end{bmatrix}$. Is the vector $\mathbf{v} = \begin{bmatrix}7\\4\\5\\11\\12\end{bmatrix}$ in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v} \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 1 & 2 & 3 & 11 \\ 2 & 1 & 4 & 12 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 2\vec{v}_3$,

so $\vec{v}$ is in span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

5. Consider the vector space $\mathbf{P}^3$. Do the polynomials $f_1(x) = 4x^3 - 2x^2 + x$, $f_2(x) = x^2 - 1$, $f_3(x) = x^2 - x$, and $f_4(x) = 3$ span $\mathbf{P}^3$? Why or why not?

$$\begin{array}{c} \\ x^3 \\ x^2 \\ x \\ \text{const.} \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 4 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix} \end{array} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the matrix of coefficients reduces to the identity matrix I_4, the polynomials are linearly independent and span $\mathbb{P}^3$.

6. Consider the 4×5 matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}.$$

(a) What is rank(A)?

(b) Find bases for col(A), null(A), and row(A).

First,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since RREF(A) has 3 pivot entries, rank$(A) = 3$.

Basis for col(A):

$$\left\{ \begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\\0\end{bmatrix} \right\}$$

Basis for row(A):

$$\left\{ \begin{bmatrix}1\\0\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \right\}$$

Basis for null(A):

$$\left\{ \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\\1\\0\end{bmatrix} \right\}$$

7. Let V be the subset of $\mathbf{P}^2$ consisting of polynomials $\mathbf{p}(x)$ with the property $\mathbf{p}(0) = 0$ and $\mathbf{p}(1) = 0$. Determine whether or not V is a subspace of $\mathbb{P}^2$. If it is not, explain why. If it is, find a basis for V.

V contains polynomials $p(x) = ax^2 + bx + c$ such that $p(0) = 0$ and $p(1) = 0$.

$p(0) = 0$ implies $p(0) = a(0)^2 + b(0) + c = c = 0$, so $p(x) = ax^2 + bx$

$p(1) = 0$ implies $p(0) = a(1)^2 + b(1) = 0$, so $b = -a$

Thus, V contains polynomials of the form $p(x) = ax^2 - ax$ for $x \in \mathbb{R}$

This means $V = \text{span}(x^2 - x)$, so V is a subspace of $\mathbb{P}^2$ with basis $\{x^2 - x\}$.

8. Suppose that the augmented matrix for a system of linear equations is

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & \frac{k}{2} \\ 1 & 0 & 8 & 1 \\ 1 & -1 & k^2 & 0 \end{array}\right]$$

(a) For which values of k, if any, does the system have no solutions?

(b) For which values of k, if any, does the system have infinitely many solutions?

(c) For which values of k, if any, does the system have a unique solution?

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & k/2 \\ 1 & 0 & 8 & 1 \\ 1 & -1 & k^2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 8 & 1 \\ 0 & 1 & 4 & k/2 \\ 1 & -1 & k^2 & 0 \end{array}\right] \xrightarrow[R_2 + R_3 \to R_3]{-R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 8 & 1 \\ 0 & 1 & 4 & \frac{k}{2} \\ 0 & 0 & k^2 - 4 & \frac{k}{2} - 1 \end{array}\right]$$

This is zero if $k = \pm 2$

If $k = -2$, then the system has no solutions.

If $k = 2$, then the system has infinitely many solutions.

Otherwise, the system has a unique solution.

9. For each of the following, the answer is *always*, *sometimes*, or *never* true.

(a) If a 3×4 matrix has three pivots after row reduction, then its columns form a linearly independent set.

Never true: the columns are vectors in $\mathbb{R}^3$, and 4 vectors in $\mathbb{R}^3$ cannot be linearly independent.

(b) The span of three (or more) vectors in $\mathbb{R}^3$ is all of $\mathbb{R}^3$.

Sometimes true

(c) If the reduced echelon form of A is I_n, then A is invertible.

Always true

(d) Let A be an $m \times n$ matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^m$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. If $\mathbf{c}$ represents a different vector in $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{c}$ has a unique solution.

Sometimes true: True for an invertible matrix.

False for the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

10. Consider the following vectors in $\mathbb{R}^3$: $\mathbf{v}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\5\\6\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

(a) The vector $\mathbf{v} = \begin{bmatrix}3\\2\\-2\end{bmatrix}$ is in span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Write $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$? Explain.

(c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly independent set or a linearly dependent set? Explain.

First,
$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v} \end{array}$$
$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 1 & 2 \\ 2 & 6 & 0 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\vec{v} = 11\vec{v}_1 - 4\vec{v}_2$, and more generally, $\vec{v} = (11-3c)\vec{v}_1 + (-4+c)\vec{v}_2 + c\vec{v}_3$ for any c in $\mathbb{R}$.

Furthermore, $3\vec{v}_1 - \vec{v}_2 = \vec{v}_3$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set that does not span $\mathbb{R}^3$.

11. Suppose $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation with matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) What are the domain and codomain of T? (Make sure to specify which is which.)

(b) Is T one-to-one? Is T onto? Explain.

T has domain $\mathbb{R}^3$ and codomain $\mathbb{R}^4$.

T cannot be onto, since $\dim(\mathbb{R}^4) > \dim(\mathbb{R}^3)$

Since $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has a pivot in each column,

T is one-to-one.

12. Consider the following matrix and its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 3 & 10 & 28 & 6 \\ 2 & 6 & 1 & -1 & 17 \\ 3 & 9 & -8 & -30 & 28 \\ -1 & -3 & 1 & 5 & 41 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of col(A), row(A), and null(A).

(b) What is the rank of A?

basis for col(A):

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 17 \\ 28 \\ 41 \end{bmatrix} \right\}$$

basis for row(A):

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

basis for null(A):

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

rank$(A) = 3$

13. Let $S = \{1 - x + x^2, 2 + x - x^2, -1 - 5x + 5x^2, x\}$ be a set of polynomials in $\mathbf{P}^2$.

(a) Is S linearly independent or dependent? Explain.

(b) Does S span $\mathbf{P}^2$? Explain.

(c) Does S form a basis for $\mathbf{P}^2$? Why or why not?

S is clearly linearly dependent since S contains 4 polynomials but $\dim(\mathbb{P}^3) = 2$.

Row-reduce the matrix whose columns are the coefficients of the polynomials:

$$\begin{array}{r} x^2: \\ x: \\ \text{const}: \end{array} \begin{bmatrix} 1 & -1 & 5 & 0 \\ -1 & 1 & -5 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since there is a pivot in each row, S spans $\mathbb{P}^2$.

Since S is linearly dependent, S is <u>not</u> a basis for $\mathbb{P}^2$.

14. Let W be the subset of all matrices A in $\mathbb{R}^{2\times 2}$ whose determinant is equal to 0. Determine whether or not W is a subspace of $\mathbb{R}^{2\times 2}$.

W is not a subspace because it is not closed under addition. For example,

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \text{and} \quad \det\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0,$$

but $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$

15. Recall that if A is a matrix, then the *transpose* of A, denoted A^T, is the matrix whose rows are the columns of A and whose columns are the rows of A (in order).

Let V be the subspace of $\mathbb{R}^{2\times 2}$ consisting of all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with the property

$$A = -A^T.$$

(a) Without finding a basis for V or (directly or indirectly) discussing free variables, explain why $0 < \dim(V) < 4$.

(b) Find a basis for V.

V contains some nonzero matrices (such as $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$), so $\dim(V) > 0$.

V does not contain all of $\mathbb{R}^{2\times 2}$ (for example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not in V), so $\dim(V) < 4$.

V contains matrices such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

This implies $a = -a$ and $d = -d$, so $a = 0$ and $d = 0$.

Also $c = -b$ and equivalently $b = -c$.

So V contains matrices of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for $b \in \mathbb{R}$

Thus, a basis for V is $\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$.

16. Find a basis for $S^\perp$ if $S = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Let A be the matrix whose rows are the given vectors, and find null(A).

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ 4 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

So null(A) contains $x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ for $x_4 \in \mathbb{R}$.

Thus, a basis for $S^\perp$ is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

17. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find the eigenvalues of A, a basis for each eigenspace, and if possible, diagonalize A.

Characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = -\lambda(2-\lambda)(3-\lambda)$$

So the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 3$

Eigenspaces:

E_0: $A - 0I = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ so a basis for E_0 is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

E_2: $A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so a basis for E_2 is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

E_3: $A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so a basis for E_3 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

A is diagonalizable: $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.