MATH 220
9 May 2025

RECALL: Vector Spaces

Examples: $\mathbb{R}^n$, $\mathbb{P}^n$, $\mathbb{R}^{m\times n}$, $\mathbb{P}$, ...

Concepts: linear combination, span, linear independence are the same in these new vector spaces as in $\mathbb{R}^n$

Subspaces: contain the zero vector, closed under vector addition and scalar multiplication

EXAMPLE: Let S consist of all polynomials $p(x)$ in $\mathbb{P}^2$ for which $p(2)=0$.

Use the definition of subspace to show that S is a subspace of $\mathbb{P}^2$.

(a) Is $\vec{0}$ in S? In $\mathbb{P}^2$, $\vec{0}$ is the polynomial $p(x)=0$

Plug in $x=2$: $p(2)=0$, so yes, $\vec{0}$ is in S.

(b) Is S closed under addition?

Suppose $p(x)$ and $q(x)$ are in S. So $p(2)=0$ and $q(2)=0$.

the number zero

We must show that $p(x)+q(x)$ is in S.

Plug in $x=2$: $p(2)+q(2) = 0+0 = 0$

So $p(x)+q(x)$ is in S, and S is closed under $+$.

(c) Is S closed under scalar multiplication?

Suppose $p(x)$ is in S and c is some scalar ($c\in\mathbb{R}$).

We must show that $c\cdot p(x)$ is in S.

Plug in $x=2$: $c\cdot p(2) = c\,0 = 0$, so $c\cdot p(x)$ is in S, and S is closed under $\cdot$

since $p(x)$ is in S

Linear Algebra – Day 37

MATH 220

1. **Erez:** It is amazing that the set S of all polynomials $p(x)$ in $\mathbf{P}^2$ for which $p(2) = 0$ is a subspace of $\mathbf{P}^2$.

   Delphine: But to show that, you had to do all three steps and that took a long time!

   Erez: Yeah, it would be nice is there was a faster way to prove S is a subspace of $\mathbf{P}^2$.

   Delphine: I have an idea. Let's see what "$p(2) = 0$" actually tells us about what the polynomial $p(x)$ has to look like.

   Erez: Oh, I see. Let's say that $p(x)$ looks like $p(x) = ax^2 + bx + c$ for some values of a, b, and c.

   Delphine: Good idea! So, if $p(x)$ happens to be in S, then $4a + 2b + c = 0$ must be true.

   Group chat: How did Delphine conclude that $4a + 2b + c = 0$ must be true?

   $$p(2) = a(2)^2 + b(2) + C = 4a + 2b + c = 0$$

   Erez: Oh, cool! That's a system of one equation. There are two free variables! Let's pretend a and b are the free variables.

   Group chat: What is the solution to the system of equation, then? Why do you think Erez decided to make a and b the free variables?

   $$c = -4a - 2b$$

   Delphine: Ohhhh...so this means that $p(x)$ really must look like $p(x) = ax^2 + bx + (-4a - 2b)$.

   Group chat: How did Delphine conclude that $p(x) = ax^2 + bx + (-4a - 2b)$?

   $$p(x) = ax^2 + bx + c$$
   $$p(x) = ax^2 + bx + (-4a - 2b) = a(x^2 - 4) + b(x - 2)$$

   Erez: That means we can write S as a span of two polynomials!

   Delphine: And, spans are subspaces!

   Group chat: How can S be written as a span of two polynomials?

   All spans are subspaces!

   $$S = \text{span}(x^2 - 4, x - 2)$$ ← set of all linear combinations of $x^2 - 4$ and $x - 2$

2. Now change S to consist of all polynomials $p(x)$ in $\mathbf{P}^2$ such that $p(2) = 1$.

   (a) Find three different polynomials that are in S.

   $p(x) = 1$, $p(x) = \frac{1}{2}x$, $p(x) = x^2 - 3$, $p(x) = x - 1$

   $p(2) = (2)^2 - 3 = 1$

   (b) Come up an example that shows how S fails to be closed under addition.

   $\underbrace{p(x) = 1, \quad q(x) = \tfrac{1}{2}x}_{\text{in } S}$ then $p(x) + q(x) = \frac{1}{2}x + 1$

   $p(2) + q(2) = \frac{1}{2}(2) + 1 = 2 \neq 1$ so not in S

   (c) Come up an example that shows how S fails to be closed under scalar multiplication.

   $p(x) = 1$ in S

   Let $c = 2$

   Then: $c \cdot p(x) = 2 \cdot 1 = 2 \neq 1$, so not in S

   (d) Does S include the zero vector?

   $p(x) = 0$ is not in S, since $p(2) = 0 \neq 1$.

NOTE: let $T = S \cup \{p(x) = 0\}$. Then T contains $\vec{0}$, but is not closed under $+$ or $\cdot$.

3. Simon: Hey Ava, look at these "vectors" from $\mathbb{R}^{2\times 2}$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Ava: Wow! These four vectors definitely span $\mathbb{R}^{2\times 2}$ *and* are linearly independent!

Group chat: Can you see why Ava says what she says?

If $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $c_1 = c_2 = c_3 = c_4 = 0$.

Simon: Oh, so this must mean these four vectors are a basis for $\mathbb{R}^{2\times 2}$.

Group chat: What does Simon mean? How would you define "basis" of a subspace S?

basis: linearly independent spanning set.

CLASS ENDED HERE

4. (a) When we studied subspaces of $\mathbb{R}^n$, how was **dimension** defined?

We'll say a bit more about basis and dimension on Monday

dimension: the number of elements in a basis

(b) Now we are studying subspaces of V. How should we define the **dimension** of a subspace S of V?

the number of elements in a basis for S

5. (a) (Review) The "standard basis" for $\mathbb{R}^4$ consists of the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This means that the dimension of $\mathbb{R}^4$ is 4.

(b) What do you think is the "standard basis" for $\mathbb{R}^{2\times 2}$? What is the dimension of $\mathbb{R}^{2\times 2}$?

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ 4

(c) What do you think is the "standard basis" for $\mathbf{P}^3$? What is the dimension of $\mathbf{P}^3$?

$\{1, x, x^2, x^3\}$ 4

(d) **Marissa (making air quotes with her fingers):** $\mathbb{R}^4$, $\mathbb{R}^{2\times 2}$, and $\mathbf{P}^3$ are all "the same."

Group chat: What does Marissa mean by this?

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow ax^3 + bx^2 + cx + d$

It's like they contain the same things, just written differently.

6. Recall that $\mathbf{P}$ is the vector space of *all* polynomials with real coefficients. What do you think is the "standard basis" for $\mathbf{P}$. What is the dimension of $\mathbf{P}$?

infinity?

$\{1, x, x^2, x^3, x^4, \ldots\}$

7. Quick theory check:

(a) Is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^{2\times 2}$? — No, it only has 3 elements

☞ Hint: count.

(b) Is $\{1 + x,\ x + x^2,\ 1 + 2x + x^2,\ x^3\}$ a basis for $\mathbf{P}^3$?

☞ There are the correct number of vectors, but look at the vectors very carefully.

No, $(1+x) + (x+x^2) = 1 + 2x + x^2$, so these polynomials are not linearly independent.