Linear Algebra – Day 36

MATH 220

1. **Cleo:** Here are three "vectors" in this crazy new vector space $\mathbf{P}^2$:

$$\mathbf{f}(x) = 2x^2 - x + 1 \qquad \mathbf{g}(x) = -5x^2 + x + 2 \qquad \mathbf{h}(x) = 13x^2 - 5x + 2$$

Milo: OK, I really want to find out whether $\mathbf{h}(x)$ is in $\text{span}(\mathbf{f}(x), \mathbf{g}(x))$.

Cleo: But, we haven't *defined* what "span" means in this crazy new world.

Milo: Why not just keep the exact same definition of *span* we had before?

Cleo: Seems convenient! I don't want to have to learn a new definition.

Milo: So, I am asking if we can write $\mathbf{h}(x)$ as a "linear combination" of $\mathbf{f}(x)$ and $\mathbf{g}(x)$.

Group chat: How is Milo using the "exact same definition of *span*" from before?

Cleo: Don't we have to define "linear combination" now, too?

Milo: Let's keep the same definition of "linear combination" we had before!

Cleo: I love your idea! We should try to write $\mathbf{h}(x) = a \cdot \mathbf{f}(x) + b \cdot \mathbf{g}(x)$ for some scalars a and b.

Group chat: How is Cleo using the "same definition" of *linear combination* as before?

Milo: I wrote out $\mathbf{h}(x) = a \cdot \mathbf{f}(x) + b \cdot \mathbf{g}(x)$. So, we must get $2a - 5b = 13$.

Group chat: Where did Milo get the equation $2a - 5b = 13$ from?

Group chat: What other equations relating a and b will have to be true?

☞ There are two more.

Group chat: Can you answer the original question? Do such a and b exist?

2. (a) Give a description of the vectors in $S = \text{span}\left(x^2, x\right)$. Can you find a vector in $\mathbf{P}^3$ that is *not* in S?

(b) Try to find a list of polynomials $p_1(x), p_2(x), \ldots, p_k(x)$ so that $\text{span}(p_1(x), p_2(x), \ldots, p_k(x))$ equals *all* of $\mathbf{P}^3$.

(c) What is the smallest number of polynomials needed to span $\mathbf{P}^3$?

3. Can you find a list of matrices A_1, A_2, $\ldots A_k$ such that $\mathbb{R}^{2\times 2} = \text{span}(A_1, A_2, \ldots, A_k)$? What is the smallest number of matrices required to span $\mathbb{R}^{2\times 2}$?

4. *Review problem*: Consider the following (familiar) vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}2\\1\\4\end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix}1\\-5\\3\end{bmatrix}$$

Are $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly independent? Explain how you know.

☞ Do not just "guess and check." Write out the DEFINITION of linear independence and apply it to these vectors.

5. Consider the following vectors in $\mathbf{P}^2$:

$$\mathbf{f}(x) = 1 + 2x^2 \qquad \mathbf{g}(x) = 2 + x + 4x^2 \qquad \mathbf{h}(x) = 1 - 5x + 3x^2$$

Are $f(x)$, $g(x)$, $h(x)$ linearly dependent or linearly independent?

☞ Use the DEFINITION of linear independence. You should end up having to do something similar to what you did previously.

6. Consider the following vectors in $\mathbb{R}^{2\times 2}$:

$$A = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \qquad B = \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}, \qquad C = \begin{bmatrix}5 & 3\\-1 & 2\end{bmatrix}$$

(a) Quick, are vectors A and B linearly dependent or linearly independent?

☞ You should be able to answer this one within 10 seconds. Why?

(b) Are A, B, and C linearly dependent or linearly independent? Explain how you know.

☞ You should NOT be able to answer this one quite as fast.