MATH 220
7 May 2025

A vector space is a collection of objects, called "vectors," that you can add together and multiply by scalars

EXAMPLES: $\mathbb{R}^n$ — Euclidean space
$\mathbb{P}^n$ — polynomials of degree $\leq n$ (variable x)
$\mathbb{R}^{m\times n}$ — all $m\times n$ matrices with real entries
$\mathbb{P}$ — all polynomials, no restriction on degree (variable x)
$x+1$, x^2-5, x^7-3x^4+2,
$x^{132}-x^2$, $x^{1000000}$, etc.

SUBSPACES: A collection of vectors S within a vector space V is a subspace of V if:
(a) The zero vector is in S
(b) S is closed under addition
(c) S is closed under scalar multiplication

EXAMPLES: $\mathbb{P}^3$ is a subspace of $\mathbb{P}^4$
$\mathbb{P}^3$ is a subspace of $\mathbb{P}$

OLD DEFINITIONS, NEW SETTING:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in some vector space V.

- A linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is any vector expressed as
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$
- The span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.
- The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span S if $S = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$.
- Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent if
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$
has <u>only</u> one solution, namely $c_1 = c_2 = \cdots = c_k = 0$.

  If there is any other solution, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly dependent.

Linear Algebra – Day 36

MATH 220

1. Cleo: Here are three "vectors" in this crazy new vector space $\mathbf{P}^2$:

$$\mathbf{f}(x) = 2x^2 - x + 1 \qquad \mathbf{g}(x) = -5x^2 + x + 2 \qquad \mathbf{h}(x) = 13x^2 - 5x + 2$$

Milo: OK, I really want to find out whether $\mathbf{h}(x)$ is in span$(\mathbf{f}(x), \mathbf{g}(x))$.

Cleo: But, we haven't *defined* what "span" means in this crazy new world.

Milo: Why not just keep the exact same definition of *span* we had before?

Cleo: Seems convenient! I don't want to have to learn a new definition.

Milo: So, I am asking if we can write $\mathbf{h}(x)$ as a "linear combination" of $\mathbf{f}(x)$ and $\mathbf{g}(x)$.

Group chat: How is Milo using the "exact same definition of *span*" from before?

span: the set of all linear combinations of some vectors

Cleo: Don't we have to define "linear combination" now, too?

Milo: Let's keep the same definition of "linear combination" we had before!

Cleo: I love your idea! We should try to write $\mathbf{h}(x) = a \cdot \mathbf{f}(x) + b \cdot \mathbf{g}(x)$ for some scalars a and b.

Group chat: How is Cleo using the "same definition" of *linear combination* as before?

linear combination: multiply each vector by a scalar, add them up

Milo: I wrote out $\mathbf{h}(x) = a \cdot \mathbf{f}(x) + b \cdot \mathbf{g}(x)$. So, we must get $2a - 5b = 13$.

Group chat: Where did Milo get the equation $2a - 5b = 13$ from?

$13x^2 - 5x + 2 = a(2x^2 - x + 1) + b(-5x^2 + x + 2)$

$13x^2 - 5x + 2 = 2ax^2 - ax + a - 5bx^2 + bx + 2b$

Group chat: What other equations relating a and b will have to be true?

There are two more.

$13 = 2a - 5b \qquad -5 = -a + b \qquad 2 = a + 2b$

$$\begin{array}{cc|c} a & b & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -5 & 13 \\ -1 & 1 & -5 \\ 1 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Group chat: Can you answer the original question? Do such a and b exist?

$a = 4, \quad b = -1$

Check: $4f(x) - 1g(x) = 4(2x^2 - x + 1) - 1(-5x^2 + x + 2)$

$= 13x^2 - 5x + 2 = h(x)$

2. (a) Give a description of the vectors in $S = \text{span}(x^2, x)$. Can you find a vector in $\mathbf{P}^3$ that is *not* in S?

polynomials $ax^2 + bx$ for some numbers a and b

examples: $5, \ x^3, \ 1, \ x^3 + 1, \ x^3 + x$

(b) Try to find a list of polynomials $p_1(x), p_2(x), \ldots, p_k(x)$ so that span$(p_1(x), p_2(x), \ldots, p_k(x))$ equals *all* of $\mathbf{P}^3$.

$x^3, x^2, x, 1$ or

$x^3 + x^2 + x + 1$

$x^2 + x + 1$

$x + 1$

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(c) What is the smallest number of polynomials needed to span $\mathbf{P}^3$?

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We did not discuss this page in class, but here are the answers, for more practice.

3. Can you find a list of matrices A_1, A_2, $\ldots A_k$ such that $\mathbb{R}^{2\times 2} = \text{span}(A_1, A_2, \ldots, A_k)$? What is the smallest number of matrices required to span $\mathbb{R}^{2\times 2}$?

4 matrices

one list is: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

4. *Review problem*: Consider the following (familiar) vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$$

Are $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly independent? Explain how you know.

Do not just "guess and check." Write out the DEFINITION of linear independence and apply it to these vectors.

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent if the eqation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.

Row reduce the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 2 & 4 & 3 & 0 \end{array}\right]$ to confirm that the only solution is the trivial solution.

5. Consider the following vectors in $\mathbf{P}^2$:

$$\mathbf{f}(x) = 1 + 2x^2 \qquad \mathbf{g}(x) = 2 + x + 4x^2 \qquad \mathbf{h}(x) = 1 - 5x + 3x^2$$

Are $f(x)$, $g(x)$, $h(x)$ linearly dependent or linearly independent?

Use the DEFINITION of linear independence. You should end up having to do something similar to what you did previously.

$f, g,$ and h are linearly independent if $c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$ has only the trivial solution.

This equation is: $c_1(1+2x^2) + c_2(2+x+4x^2) + c_3(1-5x+3x^2) = 0$

Collect like terms to obtain:
$$\begin{array}{rl} x^2: & 2c_1 + 4c_2 + 3c_3 = 0 \\ x: & 0 + c_2 - 5c_3 = 0 \\ \text{Const}: & c_1 + 2c_2 + c_3 = 0 \end{array},$$
which gives the same augmented matrix as in #4.
Thus, $f(x)$, $g(x)$, and $h(x)$ are linearly independent.

6. Consider the following vectors in $\mathbb{R}^{2\times 2}$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix}$$

(a) Quick, are vectors A and B linearly dependent or linearly independent?

You should be able to answer this one within 10 seconds. Why?

A and B are linearly independent since they have zeros in different positions

(b) Are A, B, and C linearly dependent or linearly independent? Explain how you know.

You should NOT be able to answer this one quite as fast.

Since $3A + 2B = C$, the matrices $A, B,$ and C are linearly dependent.