MATH 220
5 May 2025

VECTOR SPACES

Let V be any collection of objects (called **vectors**) such that:

- We have a way to "add" the vectors (use symbol $+$)
- We have a way to "multiply by scalars" (use symbol $\cdot$)

(Definition 7.1)

We say that V is a **vector space** if all of the following are true.
For all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V, and all scalars c and d.

(1) $\vec{u}+\vec{v}$ is also in V. ← V is closed under addition

(2) $c\cdot\vec{u}$ is also in V. ← V is closed under scalar multiplication

(3) There is some vector, named $\vec{0}$, such that $\vec{u}+\vec{0}=\vec{u}$ ← zero vector

(4) There is some vector, named $-\vec{u}$, such that $\vec{u}+(-\vec{u})=\vec{0}$. ← additive inverse

(5a) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ ← addition is commutative

(5b) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ ← addition is associative

(5c) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ ← scalar mult. distributes over vector addition

(5d) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ ← scalar mult. distributes over scalar addition

(5e) $(cd)\vec{u}=c(d\vec{u})$ ← associative property of scalar mult.

(5f) $1\cdot\vec{u}=\vec{u}$ ← 1 is the mult. identity for scalar mult.

EXAMPLE 1: $V = \mathbb{R}^2$ = collection of all $\begin{bmatrix} x \\ y \end{bmatrix}$, $x, y \in \mathbb{R}$

addition: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$

Scalar multiplication: $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$

EXAMPLE 2: $V = \mathbb{P}^2$ = collection of all polynomials of degree ≤ 2

sample "vectors": $\vec{u} = x^2 + 4$, $\vec{v} = x^2 - 7x + 3$

$\vec{w} = x - 5$ $\vec{y} = 3 = 0x^2 + 0x^1 + 3x^0$

addition: usual polynomial addition:

$$\underbrace{(a_2x^2 + a_1x + a_0)}_{\text{some generic polynomial in } \mathbb{P}^2} + \underbrace{(b_2x^2 + b_1x + b_0)}_{\text{another generic polynomial in } \mathbb{P}^2} = (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$$

Scalar multiplication: usual multiplication

$$c \cdot (a_2x^2 + a_1x + a_0) = ca_2x^2 + ca_1x + ca_0$$

Linear Algebra – Day 35

MATH 220

Here are sets of "vectors" along with definitions for "addition" and "scalar multiplication." For each, give an example of a vector. Then decide whether the set is closed under addition and scalar multiplication. Is there a "zero vector"? Do the other properties hold? If not, which fail? Finally, decide if each is a vector space.

V, $+$, and $\cdot$	Vector example	Closed under $+$?	Zero vector $\mathbf{0}$?	Closed under $\cdot$?	Other properties?	Vector space?
$\mathbb{R}^3$ with usual $+$ and $\cdot$	$\begin{bmatrix} 4 \\ -13 \\ 2.5 \end{bmatrix}$	yes	$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	yes	yes	yes
$\mathbb{R}^4$ with usual $+$ and $\cdot$	$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$	yes	$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	yes	yes	yes
$\mathbf{P}^2$: polynomials of degree ≤ 2 (variable is x), with polynomial $+$ and scalar $\cdot$	$4x^2 - 13x + 2.5$	yes	$\mathbf{0}$ is the number 0	yes	yes	yes
all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ where $x, y \geq 0$, with same $+$ and $\cdot$ as $\mathbb{R}^2$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	yes	$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	NO for example: $-1\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ not in V		NO
$\mathbb{R}^{2\times 2}$: all 2×2 matrices, with usual matrix $+$ and scalar $\cdot$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	yes	$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	yes	yes	yes
$\mathbb{R}^{3\times 2}$: all 3×2 matrices, with usual matrix $+$ and scalar $\cdot$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$	yes	$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	yes	yes	yes
all polynomials of degree *exactly* 2 (variable is x), with polynomial $+$ and scalar $\cdot$	$x^2 - 2x + 7$	NO for example: $x^2 + (-x^2) = 0$ not in V		NO for example: $0 \cdot (x^2) = 0$ not in V		NO

V, $+$, and $\cdot$	Vector example	Closed under $+$?	Zero vector $\mathbf{0}$?	Closed under $\cdot$?	Other properties?	Vector space?
upper-triangular 2×2 matrices with matrix $+$ and scalar $\cdot$	$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$	yes	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ This <u>is</u> upper triangular	yes	yes	yes
$\mathbb{R}^+$: all positive real numbers, "$\mathbf{r}+\mathbf{s}$" is done by taking $\mathbf{rs}$, "$c\cdot\mathbf{r}$" is done by taking $\mathbf{r}^c$	3.4 (or <u>any</u> positive real number)	yes If $\vec{r}>0$ and $\vec{s}>0$, then $\vec{r}+\vec{s}=\vec{r}\vec{s}>0$.	$\vec{0}=1$ For any $\vec{r}>0$, $\vec{0}+\vec{r}=1\cdot\vec{r}$ $=\vec{r}^1$ $=\vec{r}$.	yes If $\vec{r}>0$ and $c\in\mathbb{R}$, $c\cdot\vec{r}=\vec{r}^c$, which is a positive real num	yes	yes
$\mathbf{P}^n$: polynomials of degree $\leq n$ (variable is x), with polynomial $+$ and scalar $\cdot$	$3x^n+5n^{-1}+$ $\cdots+7x-3$ (degree is $\leq$ some fixed number n)	yes	$\vec{0}=0$ the number zero	yes	yes	yes
$\mathbb{R}^{m\times n}$: all $m\times n$ matrices, with usual matrix $+$ and scalar $\cdot$	$\begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & n{+}1 \\ \vdots & \vdots & \ddots & \vdots \\ m & m{+}1 & \cdots & m{+}n \end{bmatrix}$ m rows, n columns	yes	$\vec{0}=\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ the mxn zero matrix	yes	yes	yes
$\mathbf{P}$: all polynomials (variable is x) with polynomial $+$ and scalar $\cdot$	2 $5x-1$ $3x^2+x-7$ x^3+6x^2-4x+3 x^4-x^2+2 $\vdots$ polynomials of <u>any</u> degree	yes	$\vec{0}=0$ the number zero	yes	yes	yes
$\mathcal{C}$: all continuous functions (real valued functions $f(x)$), with ordinary function $+$ and scalar $\cdot$	$f(x)=x+1$ $f(x)=e^x$ $f(x)=\cos(x)+2$ etc.	yes	$\vec{0}=0$ the zero <u>function</u> $f(x)=0$	yes	yes	yes