Linear Algebra – Day 33

MATH 220

1. If $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, *why* can't $\lambda = 0$ be an eigenvalue of A?

☞ Explain without reference to the Unifying Theorem.

If T is one-to-one, then $T^{-1}(\vec{x})$ is only the zero vector, so there is no nonzero vector $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = \vec{0} = 0\vec{x}$. Therefore, $\lambda = 0$ cannot be an eigenvalue of A (by the definition of eigenvalue).

2. If $\lambda = 0$ is an eigenvalue of matrix A, *why* is A not invertible?

If $\lambda = 0$ is an eigenvalue of A, then there is some nonzero vector $\vec{x}$ such that $A\vec{x} = 0\vec{x}$. Then $A(c\vec{x}) = \vec{0}$ for every scalar multiple $c\vec{x}$ of $\vec{x}$. So the transformation $T(\vec{x}) = A\vec{x}$ is not one-to-one, and thus not invertible, which implies that A is not invertible.

3. If A is a diagonalizable matrix, how does $\det(A)$ relate to the eigenvalues of A?

☞ In fact, the relation also holds for non-diagonalizable square matrices, but this is harder to see.

If A is diagonalizable, then $A = PDP^{-1}$, where P is a diagonal matrix whose diagonal entries are $\lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of A (with multiplicity)

Then $\det(A) = \det(P)\det(D)\det(P^{-1}) = \det(P)\det(D)\dfrac{1}{\det(P)} = \det(D)$.

Thus, $\det(A) = \det(D) = \lambda_1 \lambda_2 \cdots \lambda_n$.

4. If S is a subspace and $\mathbf{u} \in S$, then what is $\text{proj}_S\, \mathbf{u}$?

If $\vec{u} \in S$, then $\text{proj}_S \vec{u} = \vec{u}$.

5. If matrix A is diagonalizable, is there more than one matrix P such that $P^{-1}AP$ is a diagonal matrix? How many different matrices P are possible?

If $\vec{x}$ is an eigenvector of A, then so is $c\vec{x}$ for any $c \neq 0$, so there are infinitely many eigenvectors of A. The columns of P are eigenvectors of A, so there are infinitely many choices for (each column of) P. Therefore, there are infinitely many possible matrices P.

6. Find all possible ranks of matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & a_{25} \\ 0 & 0 & 0 & 0 & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}.$$

Rows 1, 2, and 3 must be linearly dependent, so there are at most two linearly independent rows. Thus, the possible ranks are 0, 1, and 2

7. Let X_n be the $n \times n$ matrix that has 0s everywhere except on its two diagonals, where it has 1s. For example, here is X_5:

$$X_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Express $\mathrm{col}(X_n)$ and $\mathrm{null}(X_n)$ as spans.

$$\mathrm{col}(X_n) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}, \cdots \right\} \qquad \mathrm{null}(X_n) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix}, \cdots \right\}$$

$\frac{n}{2}$ vectors if n is even; $\frac{n+1}{2}$ vectors if n is odd

$\frac{n}{2}$ vectors if n is even $\frac{n-1}{2}$ vectors if n is odd

8. Can a matrix be diagonalizable but not invertible? Can a matrix be invertible but not diagonalizable? Explain or give examples.

Yes to both! Examples:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable (in fact, diagonal) but not invertible.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

9. For a matrix A, show that $(\mathrm{col}(A))^\perp = \mathrm{null}(A^T)$.

First, $\mathrm{col}(A) = \mathrm{row}(A^T)$.

A vector $\vec{x}$ is in $(\mathrm{row}(A^T))^\perp$ if and only if $\vec{x}$ is orthogonal to each row of A^T This occus exactly when $A^T\vec{x} = \vec{0}$, meaning $\vec{x}$ is in $\mathrm{null}(A^T)$.

Thus, $(\mathrm{col}(A))^\perp = (\mathrm{row}(A^T))^\perp = \mathrm{null}(A^T)$.