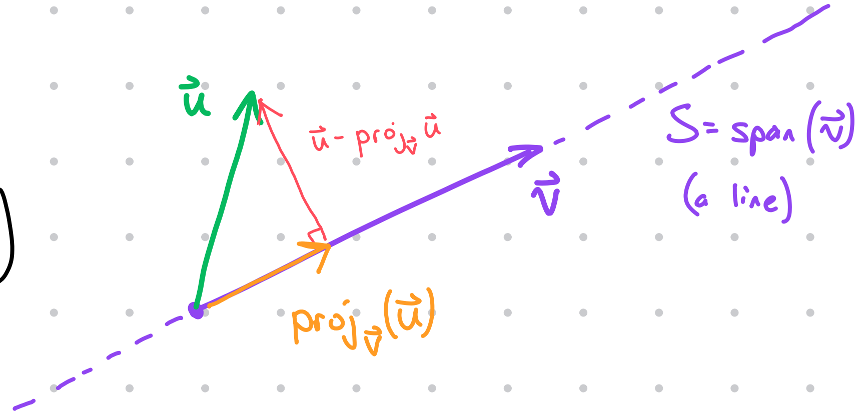
MATH 220

28 April 2025

RECALL: Projections

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\vec{u}}_{\text{parallel to } \vec{v}} + \underbrace{\left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right)}_{\text{orthogonal to } \vec{v}}$$



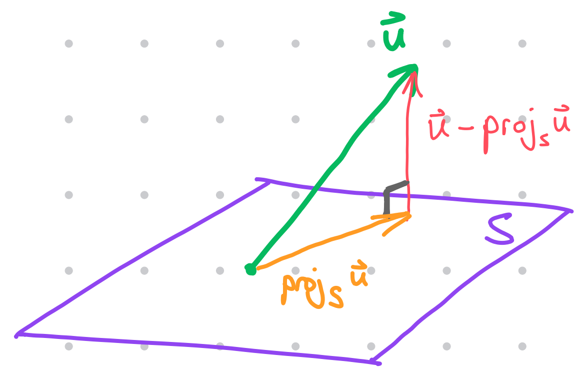
Formula: $\quad \text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$

Now: Projection onto a subspace

Let S be any subspace.

Goal: given $\vec{u}$, write:

$$\vec{u} = \underbrace{\begin{bmatrix} \text{a vector} \\ \text{in } S \end{bmatrix}}_{\text{proj}_S\vec{u}} + \underbrace{\begin{bmatrix} \text{a vector} \\ \text{in } S^\perp \end{bmatrix}}_{u - \text{proj}_S\vec{u}}$$



$$\boxed{\text{Of all the vectors in } S, \ \text{proj}_S\vec{u} \text{ is the closest to } \vec{u}}$$

Q: How do we find $\text{proj}_S\vec{u}$?

Step 1: Find a basis for S consisting of orthogonal vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$. — How? Wait for Mathematica Lab 10.

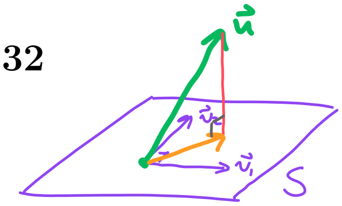
Step 2: Find projections of $\vec{u}$ onto each basis vector

$$\text{proj}_S\vec{u} = \text{proj}_{\vec{v}_1}\vec{u} + \text{proj}_{\vec{v}_2}\vec{u} + \cdots + \text{proj}_{\vec{v}_k}\vec{u}$$

$$= \frac{\vec{v}_1\cdot\vec{u}}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2\cdot\vec{u}}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \cdots + \frac{\vec{v}_k\cdot\vec{u}}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k$$

Linear Algebra – Day 32

MATH 220



1. Suppose $S = \text{span}\left(\underset{\vec{v}_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\-1\end{bmatrix}} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

$$\underset{\vec{v}_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}} \cdot \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\-1\end{bmatrix}} = 1 + 0 - 1 = 0$$

so yes

(b) Write the vector $\mathbf{u} = \begin{bmatrix}4\\3\\1\end{bmatrix}$ as the sum of $\underbrace{\text{a vector in } S}_{\text{proj}_S \vec{u}}$ and $\underbrace{\text{a vector in } S^\perp}_{\vec{u} - \text{proj}_S \vec{u}}$.

$$\text{proj}_S \vec{u} = \text{proj}_{\vec{v}_1} \vec{u} + \text{proj}_{\vec{v}_2} \vec{u}$$
$$= \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$
$$= \frac{4+6+1}{1+4+1} \begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{4+0-1}{1+0+1} \begin{bmatrix}1\\0\\-1\end{bmatrix}$$
$$= \frac{11}{6} \begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{3}{2} \begin{bmatrix}1\\0\\-1\end{bmatrix} = \frac{1}{3} \begin{bmatrix}10\\11\\1\end{bmatrix}$$

$$\begin{bmatrix}4\\3\\1\end{bmatrix} - \frac{1}{3} \begin{bmatrix}10\\11\\1\end{bmatrix} = \frac{1}{3} \begin{bmatrix}2\\-2\\2\end{bmatrix}$$

ANSWER:

$$\begin{bmatrix}4\\3\\1\end{bmatrix} = \underbrace{\frac{1}{3} \begin{bmatrix}10\\11\\1\end{bmatrix}}_{\text{in } S} + \underbrace{\frac{1}{3} \begin{bmatrix}2\\-2\\2\end{bmatrix}}_{\text{in } S^\perp}$$

2. Suppose $S = \text{span}\left(\underset{\vec{v}_1}{\begin{bmatrix}2\\-1\\3\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}2\\1\\-1\end{bmatrix}} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

Yes

(b) Write the vector $\mathbf{u} = \begin{bmatrix}2\\5\\-1\end{bmatrix}$ as the sum of a vector in S and a vector in $S^\perp$.

$$\text{proj}_S \vec{u} = \text{proj}_{\vec{v}_1} \vec{u} + \text{proj}_{\vec{v}_2} \vec{u}$$
$$= \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$
$$= \frac{4-5-3}{4+1+9} \vec{v}_1 + \frac{4+5+1}{4+1+1} \vec{v}_2$$
$$= \frac{-4}{14} \begin{bmatrix}2\\-1\\3\end{bmatrix} + \frac{10}{6} \begin{bmatrix}2\\1\\-1\end{bmatrix} = \frac{1}{21} \begin{bmatrix}58\\41\\-53\end{bmatrix}$$

$\vec{u} - \text{proj}_S \vec{u}$

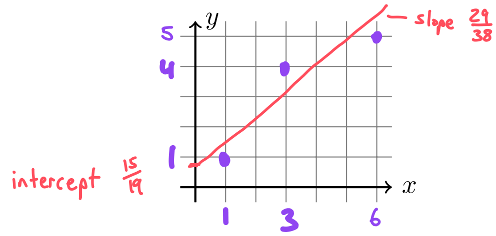
$$= \begin{bmatrix}2\\5\\-1\end{bmatrix} - \frac{1}{21} \begin{bmatrix}58\\41\\-53\end{bmatrix} = \frac{1}{21} \begin{bmatrix}-16\\64\\32\end{bmatrix}$$

Sorry, these numbers were wrong in class!

3. **Milo:** Hey Marissa! I collected some data for my big research project. So far, I have three points: $\{(1,1),(3,4),(6,5)\}$.

Marissa: A good start! It's a shame your three points are not on a common line.

Quick task: Plot the points and visually confirm they are not on the same line.



Milo: Wow, that's too bad. I was *really* hoping that a line would fit my data.

Marissa: You give up too quickly, Milo. Just pretend, for a moment, that the three data points *actually do* fit on a single line with equation $y = mx + b$.

Milo: Alright, I'll go along with your idea even though it clearly won't work.

Group chat: If Milo pretends the three points were all on the line $y = mx + b$, then what three equations need to be satisfied by b and m?

☞ One of the equations is $6m + b = 5$.

$(1,1)$: $1 = 1m + b$ $(3,4)$: $4 = 3m + b$ $(6,5)$: $5 = 6m + b$

Marissa: See? You can now write your three equations as $A\mathbf{x} = \mathbf{y}$.

Group chat: What are A, $\mathbf{x}$, and $\mathbf{y}$?

$$\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$m + b = 1$
$3m + b = 4$
$6m + b = 5$

Group chat: Is $A\mathbf{x} = \mathbf{y}$ a consistent system or an inconsistent system? Use row reduction to confirm this.

$$[A \mid y] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 1 & 4 \\ 6 & 1 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Milo: See? We can't do anything! The system has no solutions. We will never find a vector $\mathbf{x}$ where $A\mathbf{x}$ will equal $\mathbf{y}$.

Marissa: Well, that's true. We can never actually get $\mathbf{y}$. So, let's just try to find a vector to "plug in" for $\mathbf{x}$ that gets us *as close to* $\mathbf{y}$ *as possible.*

Group chat: Discuss Marissa's idea. If we can do what Marissa suggests, what are we really finding?

$\vec{y}$ is not in $\mathrm{Col}(A)$

However, $\mathrm{proj}_{\mathrm{Col}(A)} \vec{y}$ is the closest vector in $\mathrm{Col}(A)$ to $\vec{y}$.

STOP *and wait for further instructions*

4. Since there is no $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{y}$, we seek a vector $\hat{\mathbf{x}}$ so that $A\hat{\mathbf{x}}$ is *closest* to $\mathbf{y}$.

(a) Compute $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{y}$.

☞ Use Mathematica

$$\hat{x} = \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 29/38 \\ 15/19 \end{bmatrix}$$

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In[5]:= a = {{1, 1}, {3, 1}, {6, 1}}
        y = {1, 4, 5}
Out[5]= {{1, 1}, {3, 1}, {6, 1}}
Out[6]= {1, 4, 5}
In[7]:= Inverse[Transpose[a].a].Transpose[a].y
Out[7]= {29/38, 15/19}
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(b) What is the equation of the line that best fits the data points?

$$y = \frac{29}{38}x + \frac{15}{19}$$

(c) Sketch the line as best as you can on your plot above.