Linear Algebra – Day 31

MATH 220

1. (a) **Ava:** I really need to find the set H of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that are orthogonal to *both*

$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Erez: Let's make a sketch! I bet we can at least find the dimension of H.

Group chat: Try to draw this situation. What does H look like geometrically? What is the dimension of H?

(b) **Ava:** OK, that's good, but I want to write H as a span.

Delphine: First, let's see if we can write a linear system satisfied by x_1, x_2, x_3.

Group chat: What is the linear system that Delphine mentioned? How does this help you write H as a span?

☞ You'll need two dot products.

(c) **Sundar:** If $\mathbf{x}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{x}$ is actually orthogonal to *any* linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Group discussion: Is Sundar correct? Why or why not?

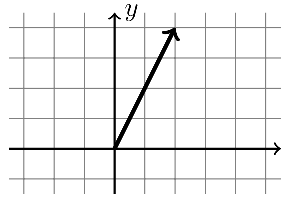
2. Let $W_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right)$, $W_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, and $W_3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

Find $W_1^\perp$, $W_2^\perp$, and $W_3^\perp$.

☞ You should be able to do all of these quickly, just by looking at the numbers.

3. Instinct check: If S is a 6-dimensional subspace of $\mathbb{R}^{10}$, what is $\dim(S^\perp)$?

4. The vector $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is drawn here.



Group discussion: How can you express $\mathbf{u}$ as a sum in the following way?

☞ Sketch these two vectors that add up to $\mathbf{u}$.

$$\mathbf{u} = \begin{bmatrix} \text{A vector parallel to} \\ \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \text{A vector orthogonal to} \\ \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix}$$

5. Josie: That last problem was super fun! Let's try another one!

Milo: OK, let's try the same thing, just modified a tiny amount. Remember subspace W_1 from problem 2? Can we express $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ as a sum in the following way?

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \left(\begin{array}{c} \text{A vector} \\ \text{in } W_1 \end{array} \right) + \left(\begin{array}{c} \text{A vector} \\ \text{in } W_1^{\perp} \end{array} \right)$$

Group chat: Try it!

6. Suppose $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

(b) Write the vector $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ as the sum of a vector in S and a vector in $S^{\perp}$.

☞ You may need to use other paper.

7. Suppose $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

(b) Explain why you cannot yet figure out how to write the vector $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ as the sum of a vector in S and a vector in $S^{\perp}$.

(c) *Challenge*: What would you have to do in order to write $\mathbf{u}$ as a sum of a vector in S and a vector in $S^{\perp}$?