MATH 220

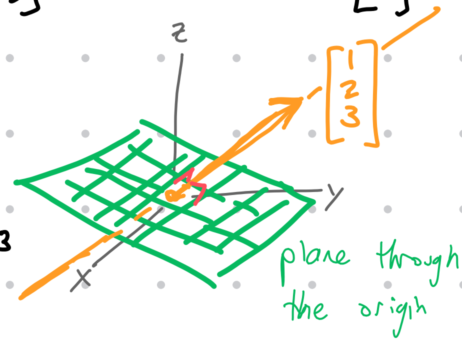
25 April 2025

$\vec{x}$ is orthogonal to $\begin{bmatrix}1\\2\\3\end{bmatrix}$

LAST TIME: H is the set of vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ such that $\vec{x} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = 0$.

Equation: $\vec{x} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = x_1 + 2x_2 + 3x_3 = 0$

Span: $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2x_2 - 3x_3\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2\\1\\0\end{bmatrix}x_2 + \begin{bmatrix}-3\\0\\1\end{bmatrix}x_3$



So, $H = \text{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right)$

In fact, $\text{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right)$ and $\text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$ are orthogonal to each other.

Generalization: If $\vec{x}$ is orthogonal to each of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$, then $\vec{x}$ is orthogonal to <u>every</u> vector in $\text{span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k)$.

The collection of all vectors $\vec{x}$ orthogonal to a given subspace S is <u>also</u> a subspace, called the orthogonal complement of S, denoted $S^\perp$.

"S perp"

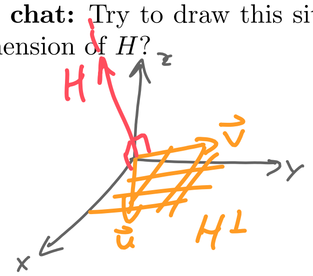
Linear Algebra – Day 31

MATH 220

1. (a) **Ava:** I really need to find the set H of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that are orthogonal to *both* $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Erez: Let's make a sketch! I bet we can at least find the dimension of H.

Group chat: Try to draw this situation. What does H look like geometrically? What is the dimension of H?



$\dim(H) = 1$

(b) **Ava:** OK, that's good, but I want to write H as a span.

Delphine: First, let's see if we can write a linear system satisfied by x_1, x_2, x_3.

Group chat: What is the linear system that Delphine mentioned? How does this help you write H as a span?

☞ You'll need two dot products.

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$

$$H = \text{span}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$$

(c) **Sundar:** If $\mathbf{x}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{x}$ is actually orthogonal to *any* linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Group discussion: Is Sundar correct? Why or why not?

$$(a\vec{u} + b\vec{v}) \cdot \vec{x} = a \underbrace{\vec{u} \cdot \vec{x}}_{0} + b \underbrace{\vec{v} \cdot x}_{0} = a0 + b0 = 0$$

We didn't do these in class

2. Let $W_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right)$, $W_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, and $W_3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

Find $W_1^\perp$, $W_2^\perp$, and $W_3^\perp$.

☞ You should be able to do all of these quickly, just by looking at the numbers.

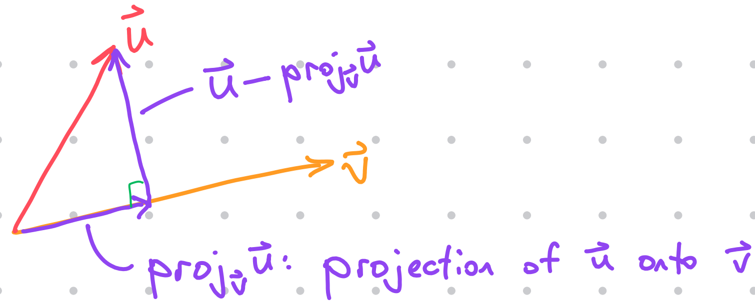
$$W_1^\perp = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \qquad W_2^\perp = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \qquad W_3^\perp = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

3. Instinct check: If S is a 6-dimensional subspace of $\mathbb{R}^{10}$, what is $\dim(S^\perp)$?

$\dim(S^\perp) = 4$, since $\dim(S) + \dim(S^\perp) = 10$

PROJECTIONS: We want to express a vector $\vec{u}$ as

$$\vec{u} = \begin{pmatrix} \text{a vector parallel} \\ \text{to } \vec{v} \end{pmatrix} + \begin{pmatrix} \text{a vector perpendicular} \\ \text{to } \vec{v} \end{pmatrix}$$



How do we calculate this projection?

Fact 1: $\mathrm{proj}_{\vec{v}}\vec{u}$ is a multiple of $\vec{v}$: $\mathrm{proj}_{\vec{v}}\vec{u} = c \cdot \vec{v}$

Fact 2: $\vec{u} - \mathrm{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$: $(\vec{u} - c\vec{v}) \cdot \vec{v} = 0$

Thus: $0 = (\vec{u} - c\vec{v}) \cdot \vec{v} = \vec{u} \cdot \vec{v} - c(\vec{v} \cdot \vec{v})$

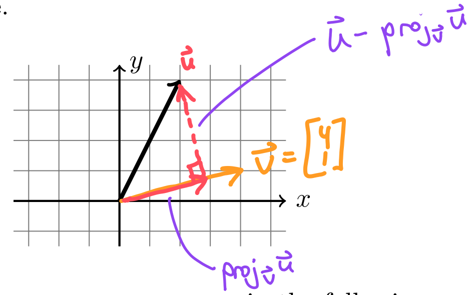
Then: $c(\vec{v} \cdot \vec{v}) = \vec{u} \cdot \vec{v}$

So: $c = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$

Therefore:

$$\boxed{\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}}$$

4. The vector $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is drawn here.



Group discussion: How can you express $\mathbf{u}$ as a sum in the following way?

Sketch these two vectors that add up to $\mathbf{u}$.

$$\mathbf{u} = \begin{bmatrix} \text{A vector parallel to} \\ \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \text{A vector orthogonal to} \\ \mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \begin{bmatrix} 48/17 \\ 12/17 \end{bmatrix} \qquad \vec{u} - \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \begin{bmatrix} -14/17 \\ 56/17 \end{bmatrix}$$

CLASS ENDED HERE

5. Josie: That last problem was super fun! Let's try another one!

Milo: OK, let's try the same thing, just modified a tiny amount. Remember subspace W_1 from problem 2? Can we express $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ as a sum in the following way?

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \left(\begin{array}{c} \text{A vector} \\ \text{in } W_1 \end{array} \right) + \left(\begin{array}{c} \text{A vector} \\ \text{in } W_1^{\perp} \end{array} \right)$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = W_1$ in $\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = W_1^{\perp}$

Group chat: Try it!

We will discuss this next week.

6. Suppose $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

(b) Write the vector $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ as the sum of a vector in S and a vector in $S^{\perp}$.

You may need to use other paper.

7. Suppose $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right)$.

(a) The two vectors you see above form a basis for S. Is this basis an orthogonal basis?

(b) Explain why you cannot yet figure out how to write the vector $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ as the sum of a vector in S and a vector in $S^{\perp}$.

(c) *Challenge*: What would you have to do in order to write $\mathbf{u}$ as a sum of a vector in S and a vector in $S^{\perp}$?